ABSTRACT ALGEBRA A
HOMEWORK 4 SOLUTIONS

Chapter 3

3-48. If $(gH)(gH) = gH$, then multiplying both sides by $g^{-1}H$ gives $gH = H$, which is impossible if $g \notin H$. This means that $g^2H = (gH)(gH) = H$, which means that $g^2 \in H$. This holds for every $g \notin H$, and since H is a subgroup of G, we know that for every $g \in H$, $g^2 \in H$. Therefore $g^2 \in H$ for all $g \in G$.

3-50. The group table looks like this:

$\cdot$	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	ab
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	a^2	a	1	a^3
ab	ab	b	a^3b	a^2b	a^3	a^2	a	1
a^2b	a^2b	ab	b	a^3b	1	a^3	a^2	a
a^3b	a^3b	a^2b	ab	b	a	1	a^3	a^2

The top half of this table is easy to compute using the fact that a generates a cyclic subgroup of order 4. The rest of the table is not hard because knowing that $ba = a^3b$ lets one move any b's in a product to the right of any a's, and because the fact that $b^2 = a^2$ then lets one get rid of extraneous copies of b. Of course, one also has to use heavily the fact that $a^4 = 1$.

Obviously this group isn't abelian. I would have said that there are 5 proper subgroups, one of which is the trivial subgroup $\{e\}$. The others are

$$\{e, a, a^2, a^3\}$$
$$\{e, b, a^2, a^2b\}$$
$$\{e, ab, a^2, a^3b\}$$
$$\{e, a^2\}.$$

The first 3 of these are obviously normal, since they have index 2. They are the cyclic subgroups generated by any of the elements of Q except e and a^2.

Since both e and a^2 commute with every element of Q (look at the group table), $\{e, a^2\}$ must be normal in Q. Further, any subgroup built by starting with $\{e, a^2\}$ and adding additional elements must contain one of the 6 elements of order 4; so it must either be one of the three subgroups of order 4, or all of Q.

All this (and practically everything else about Q) is simpler to think about if we regard Q as consisting of the elements ± 1, $\pm i$, $\pm j$ and $\pm k$ where $i^2 = j^2 = k^2 = -1$

and

$$\begin{aligned} ij &= -ji = k \\ jk &= -kj = i \\ ki &= -ik = j. \end{aligned}$$

3-51. You can pretty much do this just by squinting at the group table for the quaternion group. All the occurrences of b are in the upper right and lower left quadrants of the table. The group table is therefore

	$\langle a \rangle$	$\langle a \rangle b$
$\langle a \rangle$	$\langle a \rangle$	$\langle a \rangle b$
$\langle a \rangle b$	$\langle a \rangle b$	$\langle a \rangle$

Of course, the fact that the subgroup $\langle a \rangle$ is normal in the quaternion group means that it is harmless to put the b on the left above; but putting it on the right seems more consistent with the way we have written the multiplication table.

3-52. What's there to say? Obviously $V = \langle v \rangle$ is abelian, but as noted at the top of page 74, it isn't normal.

3-53. We already know that $V \ntriangleleft G$. That $V \triangleleft K$ and that $K \triangleleft G$ are easy because $[K : V] = [G : K] = 2$.

Chapter 4

4-3. Well, if it's really a homomorphism then the kernel is easy to find: it's the set of four elements $\{1, r, s, t\}$ that are the only elements left that the homomorphism could take to the identity.

4-4. Here's a fairly economical way to do this one:

One of the left cosets of K is $K = \{1, r, s, t\}$ itself. Another is $aK = \{a, c, b^2, d^2\}$. The left cosets form a disjoint partition of $\mathcal{A}_4$; so the remaining left coset is $\{a^2, b, c^2, d\}$.

Similarly, the right cosets of K are K itself, $Ka = \{a, c, b^2, d^2\}$, and $\{a^2, b, c^2, d\}$.

One of our equivalent definitions for normal subgroups was that they are subgroups for which every left coset is a right coset. That's satisfied here, so $K \triangleleft \mathcal{A}_4$.

4-5. I assume that what they mean by finding $\mathcal{A}_4/K$ is computing the group table of $\mathcal{A}_4/K$, which is shown below.

	K	aK	bk
K	K	aK	bK
aK	aK	bK	K
bk	bK	K	aK

This is the same as the group table of the cyclic group of order 3 if we replace K with e, aK with T, and bK with T^2:

	e	T	T^2
e	e	T	T^2
T	T	T^2	e
T^2	T^2	e	T

Thus, $\mathcal{A}_4/K \simeq \langle T \rangle$.

4-6. The homomorphism has to take e and a to 0 and it has to take b and c to 1, assuming the cyclic group with 2 elements is being written as $(\mathbb{Z}_2, +)$. This is a homomorphism because $\{e, a\} \lhd K_4$ by Theorem 3-11.

4-7. This time e and a^2 map to 0, and a and a^3 map to 1. Again, it has to be a homomorphism because the kernel is a subgroup of index 2, which means that it's normal by Theorem 3-11.

4-8. This one was a little tricky, since if $a + \langle n \rangle$ and $b + \langle n \rangle$ are two cosets, then the product

$$(a + \langle n \rangle)(b + \langle n \rangle) = \{\, xy : x \in a + \langle n \rangle,\ y \in b + \langle n \rangle \,\}$$

is not a coset. We therefore have to begin by defining what the product of two cosets should be. The natural hope would be that we could define the product as

$$(a + \langle n \rangle)(b + \langle n \rangle) = ab + \langle n \rangle. \tag{1}$$

If this product is well-defined, then the mapping $\mathbb{Z} \to \mathbb{Z}/\langle n \rangle$ defined by $\phi(a) = a + \langle n \rangle$ is automatically a homomorphism under multiplication, since that's what equation (1) says.

So what we need to prove is that if $a + \langle n \rangle = a' + \langle n \rangle$ and $b + \langle n \rangle = b' + \langle n \rangle$, then $ab + \langle n \rangle = a'b' + \langle n \rangle$. This is easy, though. If $a + \langle n \rangle = a' + \langle n \rangle$ and $b + \langle n \rangle = b' + \langle n \rangle$, then

$$a' = a + n_1$$
$$b' = b + n_2$$

for some n_1 and n_2 in $\langle n \rangle$. This means that

$$a' = a + nx$$
$$b' = b + ny$$

for some x and y in $\mathbb{Z}$. But then

$$\begin{aligned}
a'b' + \langle n \rangle &= (a + nx)(b + ny) + \langle n \rangle \\
&= (ab + n(ay + bx + nxy)) + \langle n \rangle \\
&= (ab + nt) + \langle n \rangle \\
&= \{ab + nt + nz : z \in \mathbb{Z}\} \\
&= \{ab + n(t + z) : z \in \mathbb{Z}\} \\
&= \{ab + nq : q \in \mathbb{Z}\} \\
&= ab + \langle n \rangle,
\end{aligned}$$

where $t = ay + bx + nxy$ and where we have used the fact that as z ranges over all elements of $\mathbb{Z}$, so does $q = t + z$.