

ABSTRACT ALGEBRA A
HOMEWORK 5 SOLUTIONS

CHAPTER 5

- 5-3.** Well, yes, the subgroup $\{1, r, s, t\}$ we worked with in the previous chapter. A more illuminating way to write the elements of this subgroup is as

$$\begin{aligned} &e \\ &(12)(34) \\ &(13)(24) \\ &(14)(23). \end{aligned}$$

- 5-4.** No, it doesn't. If it did have a subgroup H of order 6, then by Theorem 5-3, H would have to contain both an element of order 2 and an element of order 3. Since every element of \mathcal{A}_4 other than the identity is either a 3-cycle or a product of 2 2-cycles, this means that H would have to contain a 3-cycle (of order 3) and a product of 2 2-cycles (of order 2).

Suppose, therefore, that H contains (123) . (If not, just rename the objects being permuted so that $(123) \in H$.) Suppose also that $(12)(34) \in H$. Then H also contains the elements

$$\begin{aligned} (132) &= (123)^{-1} \\ (134) &= (123)(12)(34) \\ (143) &= (134)^{-1} \\ (14)(23) &= (123)(143). \end{aligned}$$

At this point, H already has order at least 7, contradicting our hope that we could build a subgroup of order 6.

Could we get a subgroup with 6 elements by adding either $(14)(23)$ or $(13)(24)$ to H in place of $(12)(34)$? No, because either of these alternatives consists of a transposition interchanging two adjacent elements of the cycle (123) and a second transposition interchanging 4 and the remaining element of the 3-cycle (123) , which means that by symmetry, we could do just the same argument we did above to get too many elements in H .

If you don't buy this argument from symmetry, just do the calculation directly. It isn't hard.

- 5-9.** The first part of this is just a calculation:

$$(234)^{-1}(123456)(234)(123456)^{-1} = (432)(345) = (245).$$

(The first equality comes from what we know about conjugating cycles by arbitrary permutations, or just by direct computation.)

Now suppose σ is a permutation moving a minimal number of elements and containing some cycle of length 4 or longer, possibly together with some other

cycles. By renaming the letters if necessary, we can write

$$\sigma = (123 \dots x)\tau,$$

where $x \geq 4$ and where τ moves only numbers $y > x$.

Our normal subgroup containing σ would then also contain

$$(1) \quad (234)^{-1}\sigma(234)\sigma^{-1} = (234)^{-1}(123 \dots x)\tau(234)\tau^{-1}(123 \dots x)^{-1}$$

We know from studying conjugates that $\tau(234)\tau^{-1} = (\tau(2)\tau(3)\tau(4)) = (234)$, since τ fixes all $y \leq 4 \leq x$. Thus, the permutation in (1) is

$$(234)^{-1}(123 \dots x)(234)(123 \dots x)^{-1} = (432)(345) = (245),$$

as shown above. This obviously moves fewer elements than the at least 4 elements moved by σ , which contradicts the assumption that σ is minimal.

5-10. Again, the beginning of this is just a simple computation:

$$\begin{aligned} & (126)^{-1}(1234)(5678)(126)[(1234)(5678)]^{-1} \\ & = (621)(237) = (16237). \end{aligned}$$

The more general calculation, which I leave you to verify, shows that if

$$\sigma = (12 \dots k)(k, k+1, \dots, 2k)$$

is a product of two k -cycles, then

$$(2) \quad (1, 2, k+2)^{-1}\sigma(1, 2, k+2)\sigma^{-1} = (1, k+2, 2, 3, k+3)$$

is a 5-cycle, which again moves fewer elements than the original σ . This holds as long as $k \geq 3$. Finally, if σ contains more than these two cycles, then an argument just like that given in Problem 9 shows that the remaining cycles do not change the product in equation (2); so that (2) continues to give us an element moving fewer elements than does σ .

5-11. This one is purely computational, but it actually takes two steps. First, do what the Maxfields suggest to show that

$$(321)(12)(34) \cdots (2k-1, 2k)(123)(2k-1, 2k) \cdots (34)(12) = (14)(23).$$

This shows that if N contains a product of disjoint transpositions, then it must contain a product of 2 disjoint transpositions. But $\sigma = (14)(23)$ cannot move the minimal number of elements moved by elements of N , since a small modification of the Maxfields' proof of *iii* for \mathcal{A}_5 shows that if $\sigma = (14)(23) \in N$, then so is

$$(145)^{-1}\sigma(145)\sigma^{-1} = (541)(14)(23)(145)(23)(14) = (145).$$

which moves fewer elements than does σ .

5-13. The proof for \mathcal{A}_5 falls conceptually into these steps:

- (1) If $\sigma \in N \triangleleft \mathcal{A}_5$ moves as few elements as possible, then every cycle in σ has the same length.
- (2) Since σ is even, σ consists either of a 5-cycle, a product of two 2-cycles, or a 3-cycle.
- (3) If σ is a 5-cycle, then N also contains a 3-cycle which moves fewer elements than does σ , so σ isn't minimal. Thus, σ can't be a 5-cycle.

- (4) If σ is a product of 2 disjoint transpositions, then N also contains a 3-cycle which moves fewer elements than does σ , so σ isn't minimal. Thus, σ can't be a product of 2 disjoint transpositions. Thus, σ must be a 3-cycle.
- (5) If $N \triangleleft \mathcal{A}_5$ contains a 3-cycle, then it must contain every 3-cycle.
- (6) If $N \triangleleft \mathcal{A}_5$ contains every 3-cycle, then $N = \mathcal{A}_5$.

The proof of step (1) works without modification in every \mathcal{A}_n .

Step (2) is still true in \mathcal{A}_4 , except that obviously σ can't consist of a 5-cycle since there are only 4 possible elements to permute.

Step (3) is therefore irrelevant, or perhaps, vacuously true.

Step (4) is false in \mathcal{A}_4 , since the normal subgroup of \mathcal{A}_4 we have seen in previous problems consists of the permutations e , $(12)(34)$, $(13)(24)$, and $(14)(23)$. It probably isn't surprising that this step should break, since the Maxfields' proof in \mathcal{A}_5 relied on using their formula (1) with $a = (125)$, which we can't do in \mathcal{A}_4 because we don't have the element 5 available.

We could really stop here, but it might be worth going on and convincing yourself that the proofs we gave for steps (5) and (6) actually work just fine in \mathcal{A}_4 . Thus, the only bit of the proof that fails is step (3).