CALCULUS A HOMEWORK 1 SOLUTIONS

Section 1.1

2. **(a)** $f(-4) = -2$; $g(3) = 4$.
(b) $f(x) = g(x)$ when $x = \pm 2$.
(c) $f(x) = -1$ when $x = -3$ and for $x = 4$.
(d) f is decreasing for $0 \le x \le 4$, that is, in the interval $[0, 4]$.
(e) The domain of f is $[-4, 4]$ and its range is $[-2, 3]$.
(f) The domain of g is $[-4, 3]$ and its range is $[1/2, 4]$.

5. This is not a function because it fails the vertical line test at many points (for instance, at $x = 0$).

6. This is a function. Its domain is $[-2, 2]$ and its range is $[-1, 2]$.

7. This is also a function. Its domain is $[-3, 2]$ and its range is

$$[-3, -2) \cup [-1, 3] = \{\, x : -3 \le x < -2 \text{ or } -1 \le x \le 3 \,\}.$$

8. This is not a function because it fails the vertical line test at every integer.

11. I would expect the temperature to be a constant 0° C until the ice melts. Then the temperature should begin to increase toward room temperature. If you know Newton's law of cooling, you would expect the warming to slow down, so that the temperature approaches room temperature only asymptotically. The sketch is shown in Figure 1. If the cold water doesn't start out at 0° C, then there will be an initial segment where the temperature drops down to freezing (actually approaching it but never quite reaching it).

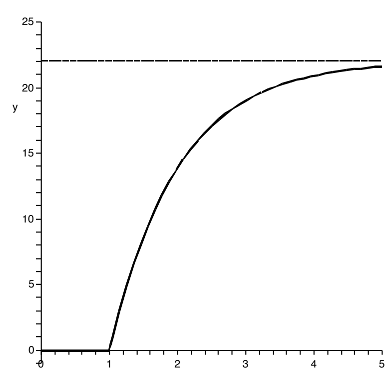

Figure 1. The temperature of the warming glass.

19. Assuming the week starts on a Sunday, the sketch is shown in Figure 2.

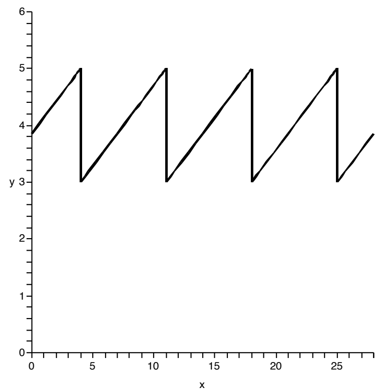

FIGURE 2. The height of the grass.

23. If $f(x) = 3x^2 - x + 2$, then

$$\begin{aligned}
f(2) &= 3(2^2) - 2 + 2 = 12 \\
f(-2) &= 3(-2)^2 - (-2) + 2 = 16 \\
f(a) &= 3a^2 - a + 2 \\
f(-a) &= 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2 \\
f(a+1) &= 3(a+1)^2 - (a+1) + 2 = 3a^2 + 5a + 4 \\
2f(a) &= 2(3a^2 - a + 2) = 6a^2 - 2a + 4 \\
f(2a) &= 3(2a)^2 - 2a + 2 = 12a^2 - 2a + 2 \\
f(a^2) &= 3(a^2)^2 - a^2 + 2 = 3a^4 - a^2 + 2 \\
[f(a)]^2 &= (3a^2 - a + 2)^2 = 9a^4 - 6a^3 + 13a^2 - 4a + 4 \\
f(a+h) &= 3(a+h)^2 - (a+h) + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.
\end{aligned}$$

24. When the radius is r, the volume is $\frac{4}{3}\pi r^3$. When the radius is $r + 1$, the volume is $\frac{4}{3}\pi(r+1)^3$. The volume of air required to get from radius r to radius $r + 1$ is therefore

$$\tfrac{4}{3}\pi(r+1)^3 - \tfrac{4}{3}\pi r^3,$$

which could be simplified in various ways if you chose to do so.

26. If $f(x) = x^3$, then

$$\begin{aligned}
\frac{f(a+h) - f(a)}{h} &= \frac{(a+h)^3 - a^3}{h} = \frac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} \\
&= \frac{3a^2h + 3ah^2 + h^3}{h} = 3a^2 + 3ah + h^2.
\end{aligned}$$

27. If $f(x) = 1/x$, then

$$\frac{f(x)-f(a)}{x-a} = \frac{\frac{1}{x}-\frac{1}{a}}{x-a} = \frac{\frac{a-x}{ax}}{x-a} = -\frac{1}{ax}$$

29. The function $f(x) = (x+4)/(x^2-9)$ is defined except when the denominator is 0. Its domain is therefore

$$\{\, x : x \neq \pm 3 \,\} = (-\infty, -3) \cup (-3, 3) \cup (3, +\infty).$$

32. Fr the function $g(t) = \sqrt{3-t} - \sqrt{2+t}$ to be defined, both square roots have to make sense. We therefore need both $3 - t \geq 0$ and $2 + t \geq 0$; i.e., we need $t \leq 3$ and $t \geq -2$. The domain is therefore $[-2, 3]$.

59. The volume is clearly length $\times$ width $\times$ depth. Looking at the picture, we see that the depth is x, the length is $20 - 2x$, and the width is $12 - 2x$. The volume is therefore $V(x) = x(20-2x)(12-2x)$. You can obviously multiply this out if you see fit.

Section 1.2

2. **(a)** π^x is an exponential function.
(b) x^π is a power function.
(c) $x^2(2-x^3)$ is a polynomial function of degree 5.
(d) $\tan t - \cos t$ is naturally a trigonometric function.
(e) $s/(1+s)$ is a rational function.
(f) All one can say about

$$\frac{\sqrt{x^3-1}}{1+\sqrt[3]{x}}$$

is that it is an algebraic function.

Technically, one could say a bit more, since every power function is a polynomial, every polynimial is a rational function, and every rational function is an algebraic function.

4. **(a)** $G(x) = 3x$, since that's the only straight line.
(b) $f(x) = 3^x$, since an exponential function like that should be assymptotic to the negative x-axis and should grow rapidly for positive x.
(c) $F(x) = x^3$ because, I don't know—that's what x^3 looks like. OK, because x^3 should grow faster and faster as x increases, and should grow negative faster and faster as x increases in the negative direction.
(d) $g(x) = \sqrt[3]{x}$ because that's what's left. Also because it looks like the inverse function of x^3. Also because that's how fractional powers grow: more and more slowly as x moves away from 0.

12. The sketch is shown in Figure 3.

The y-intercept of this curve is the number of spaces rented if the cost is 0. The x-intercept is the price at which no one is willing to rent a space. The slope (which will be measured in spaces/dollar, right?) is the number of spaces he loses for every additional dollar he charges.

A question for you, a very calculus-y question of obvious economic importance: what is the price that maximizes the manager's revenue?

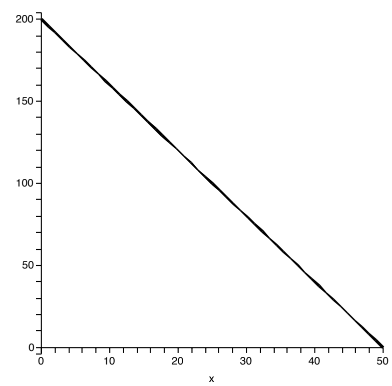

FIGURE 3. The flealine.

15. (a) The function we are looking for is a line going through the points $(113, 70)$ and $(173, 80)$. Its slope is therefore $(80 - 70)/(173 - 113) = \frac{1}{6}$. The line of this slope through the first point in question has equation

$$y - 70 = \tfrac{1}{6}(x - 113)$$
$$y = \tfrac{1}{6}x + \tfrac{307}{6}.$$

(b) The slope of $\frac{1}{6}$ represents the numbers of degrees increase in temperature corresponding to 1 additional chirp per minute.

(c) Well, if $x = 150$, then

$$y = \tfrac{1}{6}(150) + \tfrac{307}{6} = \tfrac{457}{6} \doteq 76.17^\circ.$$

18. (a) If the function is linear, then it's a straight line passing through the points $(480, 380)$ and $(800, 460)$. The slope of this line is $(460 - 380)/(800 - 480) = \frac{1}{4}$. Its equation is then

$$y - 380 = \tfrac{1}{4}(x - 480)$$
$$y = \tfrac{1}{4}x + 260.$$

(b) The cost of driving 1500 miles per month would be $\frac{1}{4}(1500) + 260 = \$635$.

(c) The sketch is shown in Figure 4. The slope is the additional cost of each additional mile driven.

(d) The y-intercept is the cost of owning the car if she doesn't drive it at all.

(e) You would expect the costs to include a piece that is independent of how far the car is driven: insurance, registration, prhaps car payments if she unwisely borrowed money to buy it, etc. These costs look like b. There will also be costs like fuel, oil, repairs, etc. that are proportional to the number of miles driven: drive twice as far, and you buy twice as much gas. These costs will look like mx. It seems like these are all the costs; in any case, it's hard to think of costs proportional to, say, the square of the mileage driven. So we're left with total costs of $y = mx + b$.

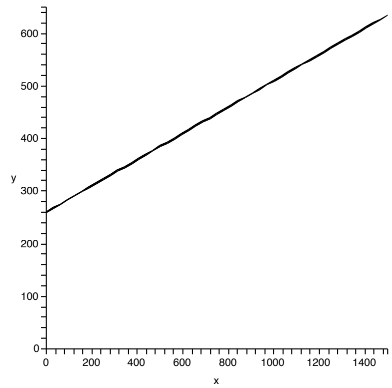

FIGURE 4. The cost of car ownership.

26. **(a)** We're being asked to find constants C and r such that $T = Cd^r$. For the earth, $T = d = 1$, which means $1 = C \cdot 1^r = C$. That makes things a lot simpler: we want an r so that $T = d^r$.

How we find this r depends on how sophisticated we want to be. We could just try different values of r until we hit one that works. We could also do some algebra. For Neptune, for instance, the planet for which we have the most precise values of T and d, we want

$$T = d^r$$
$$164.784 = 30.086^r$$
$$r = \log_{30.086} 164.784 = \frac{\log_{10} 164.784}{\log_{10} 30.086} = \frac{2.39506}{1.59667} = 1.49957.$$

This suggests very strongly that we ought to have $r = 3/2$. For all the planets, this gives very close agreement with the data in the table. (In fact, the worst agreement is with Neptune.) So in every case, the year is very nearly the distance to the $3/2$ power.

(b) Well, if year = distance$^{3/2}$, then year^2 = distance^3, right? The fact that we got equality instead of proportionality is because we scaled things so that the earth is both at distance 1 and at period 1. Measuring distance in furlongs and time in months would introduce a constant of proportionality.