

## CALCULUS A HOMEWORK 2 SOLUTIONS

### SECTION 1.3

1. (a)  $y = f(x) + 3$ .  
(b)  $y = f(x) - 3$ .  
(c)  $y = f(x - 3)$ .  
(d)  $y = f(x + 3)$ .  
(e)  $y = -f(x)$ .  
(f)  $y = f(-x)$ .  
(g)  $y = 3f(x)$ .  
(h)  $y = f(x)/3$ .
3. (a)=3, (b)=1, (c)=4, (d)=5, (e)=2. It's hard to give reasons beyond the obvious ones.
6. The original graph seems to be doubled in height and slid right a distance 2; so it would be  $y = 2f(x - 2)$ .
7. This seems to be turned over, slid left 4, and slid down 1; so how about  $y = -f(x + 4) - 1$ ?
27. (a) To the right of the  $y$ -axis, the graph of  $y = f(|x|)$  and the graph of  $y = f(x)$  look exactly alike; to get the part of the graph of  $y = f(|x|)$  to the left of the  $y$ -axis, take the part of the graph of  $y = f(x)$  to the right of the  $y$ -axis and flip it over the  $y$ -axis to produce an even function.  
(b) The graph of  $y = \sin |x|$  is in Figure 1.  
(c) The graph of  $y = \sqrt{|x|}$  is in Figure 2.

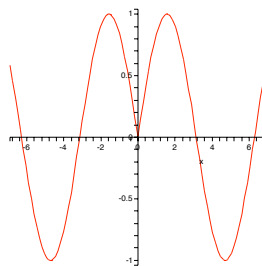
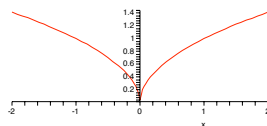
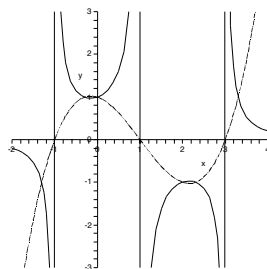


FIGURE 1.  $y = \sin |x|$ .

28. A sketch is in Figure 3. We pretty well walked through the procedure in class, focusing on the zeros of  $f$ , which correspond to the vertical asymptotes of  $1/f$ . We

FIGURE 2.  $y = \sqrt{|x|}$ .

also looked at places where  $f = 1/f = \pm 1$  and places where  $f \rightarrow \pm\infty$ , so that  $1/f \rightarrow 0$ .

FIGURE 3. Problem 28:  $y = f(x)$  and  $y = 1/f(x)$ .

- 32.**  $(f \circ g)(x) = (x^2 + 3x + 4) - 2 = x^2 + 3x + 2$ .  $(g \circ f)(x) = (x - 2)^2 + 3(x - 2) + 4 = x^2 - x + 2$ .  $(f \circ f)(x) = (x - 2) - 2 = x - 4$ .  $(g \circ g)(x) = (x^2 + 3x + 4)^2 + 3(x^2 + 3x + 4) + 4 = x^4 + 6x^3 + 20x^2 + 33x + 32$ . All these functions have the entire real line as their domains.
- 33.**  $(f \circ g)(x) = 1 - 3 \cos x$ .  $(g \circ f)(x) = \cos(1 - 3x)$ .  $(f \circ f)(x) = 1 - 3(1 - 3x) = 9x - 2$ .  $(g \circ g)(x) = \cos(\cos x)$ . All these functions have the entire real line as their domains.
- 34.**  $(f \circ g)(x) = \sqrt{\sqrt[3]{1-x}} = \sqrt[6]{1-x}$ .  $(g \circ f)(x) = \sqrt[3]{1-\sqrt{x}}$ .  $(f \circ f)(x) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$ .  $(g \circ g)(x) = \sqrt[3]{1-\sqrt[3]{1-x}}$ .

To find the domains of these functions, all we have to remember is that we can take odd roots of any number, but even roots only of non-negative numbers. Thus, the domain of  $f \circ g$  is  $(-\infty, 1]$ , the domains of  $g \circ f$  and  $f \circ f$  are both  $[0, +\infty)$ , and the domain of  $g \circ g$  is  $\mathbb{R} = (-\infty, +\infty)$ .

35. There's a bit more algebra here.

$$\begin{aligned}(f \circ g)(x) &= \frac{x+1}{x+2} + \frac{x+2}{x+1} = \frac{2x^2 + 6x + 5}{(x+1)(x+2)} \\(g \circ f)(x) &= \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2} = \frac{x^2 + x + 1}{(x+1)^2} \\(f \circ f)(x) &= \left(x + \frac{1}{x}\right) + \frac{1}{x + \frac{1}{x}} = \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)} \\(g \circ g)(x) &= \frac{\frac{x+1}{x+2} + 1}{\frac{x+1}{x+2} + 2} = \frac{2x+3}{3x+5}.\end{aligned}$$

The only thing that prevents these functions from having all real numbers for their domains is that you can't divide by 0. You therefore expect that the domain of  $f \circ g$  will be  $\{x : x \neq -1 \text{ and } x \neq -2\}$ , the domain of  $g \circ f$  will be  $\{x : x \neq -1\}$ , the domain of  $f \circ f$  will be  $\{x : x \neq 0\}$ , and the domain of  $g \circ g$  will be  $\{x : x \neq -5/3\}$ . In fact, though, the situation is a bit more complicated than this.

To see why, look again at  $(g \circ f)(x)$ . Obviously it is undefined when  $x = -1$ , but since

$$(g \circ f)(x) = g(f(x)) = \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2},$$

it is also undefined when  $f(x) = x + \frac{1}{x}$  is undefined, i.e., when  $x = 0$ . The domain of  $g \circ f$  is therefore  $\{x : x \neq -1 \text{ and } x \neq 0\}$ . For the same reason,  $(g \circ g)(x)$  is undefined not only at  $x = -5/3$ , but also where  $g(x)$  is undefined, i.e., at  $x = -1$  and  $x = -2$ . Its domain is therefore  $\{x : x \neq -1 \text{ and } x \neq -2 \text{ and } x \neq -5/3\}$ . On the other hand, we were right about the domains of  $f \circ f$  and  $f \circ g$ .

41.  $f(x) = x^4$  and  $g(x) = 2x + x^2$ .

Of course, there are infinitely many other less obvious answers one could give to this question. Perhaps  $f(x) = x^2$  and  $g(x) = (2x + x^2)^2$ . Perhaps  $f(x) = (2x + x^2)^4$  and  $g(x) = x$ . Perhaps  $f(x) = \tan x$  and  $g(x) = \arctan((2x + x^2)^4)$ .

In the next three problems, I'll give only what I think is the obvious answer.

42.  $f(x) = x^2$  and  $g(x) = \cos x$ .

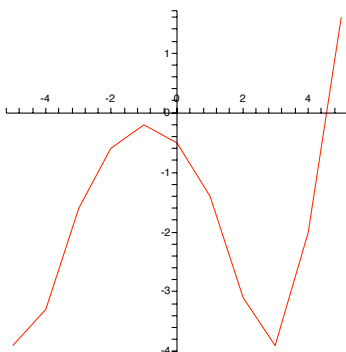
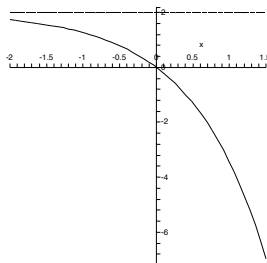
43.  $f(x) = \frac{x}{1+x}$  and  $g(x) = \sqrt[3]{x}$ .

44.  $f(x) = \sqrt[3]{x}$  and  $g(x) = \frac{x}{1+x}$ .

52. Estimate like this:  $f(g(-5)) = f(-0.2) = -3.9$ . Similarly, I would approximate  $f(g(-4)) = -3.3$ ,  $f(g(-3)) = -1.6$ ,  $f(g(-2)) = -0.6$ ,  $f(g(-1)) = -0.2$ ,  $f(g(0)) = -0.5$ ,  $f(g(1)) = -1.4$ ,  $f(g(2)) = -3.1$ ,  $f(g(3)) = -3.9$ ,  $f(g(4)) = -2$ ,  $f(g(5)) = 1.6$ . A graph obtained by just connecting up these points is shown in Figure 4. The real graph is presumably somewhat smoother than this.

## SECTION 1.5

16. To get a rough sketch, start with the graph of  $y = e^x$ . Reflect it across the  $x$ -axis to get  $y = -e^x$ . Move it one unit up to get  $y = 1 - e^x$ . Finally, stretch vertically by a factor of 2 to get  $y = 2(1 - e^x)$  (Of course, there are other ways to get the curve.) The rough graph is in Figure 5.

FIGURE 4. Problem 52:  $(f \circ g)(x)$ .FIGURE 5. Problem 16:  $y = 2(1 - e^x)$ .

17. I assume we are to start fresh in each part of the problem. If this is all part of a single process, then you'll get different and more complicated answers.

(a)  $y = e^x - 2$ . (b)  $y = e^{x-2}$ . (c)  $y = -e^x$ . (d)  $y = e^{-x}$ . (e)  $y = -e^{-x}$ .

22. From the graph, we know that

$$Ca^1 = Ca = \frac{4}{3}$$

$$Ca^{-1} = \frac{C}{a} = 3.$$

The second equation means that  $C = 3a$ . Plugging this into the first equation gives  $3a^2 = 4/3$ , so that  $a^2 = 4/9$ , and  $a = \pm 2/3$ . It therefore appears that we have two solutions:  $a = 2/3$ ,  $C = 2$ , and  $a = -2/3$ ,  $C = -2$ . The equation  $f(x) = Ca^x$  is, however, defined for all  $x$  only if  $a \geq 0$ . (If  $a < 0$ , then  $Ca^{1/2}$  does not exist, for instance.) Since the picture seems to suggest that  $f(x)$  is defined for all  $x$ , this means that we can rule out the second solution, and that  $f(x) = 2(2/3)^x$ .

23. If  $f(x) = 5^x$ , then  $f(x+h) = 5^{x+h} = 5^x 5^h$ . Thus,

$$\frac{f(x+h) - f(x)}{h} = \frac{5^{x+h} - 5^x}{h} = \frac{5^x 5^h - 5^x}{h} = \frac{5^x(5^h - 1)}{h} = 5^x \left( \frac{5^h - 1}{h} \right).$$

24. Assuming the month has 30 days, the second payment schedule gives me

$$1 + 2 + 2^2 + 2^3 + \dots + 2^{29} = 2^{30} - 1 = 1073741823 \text{ cents} = \$10,737,418.23.$$

Obviously, I would prefer this scheme. If the month has 31 days, I'd like it even better. Even if the month has only 28 days, the second scheme gives me

$$1 + 2 + 2^2 + 2^3 + \dots + 2^{27} = 2^{28} - 1 = 268435455 \text{ cents} = \$2,684,354.55.$$

Only for months of 26 or fewer days would the first scheme be preferable.

26. There are two points of intersection: one at  $x = 1.764921915$ , which *Maple* finds easily, and one at  $x = 5$ , which humans find by looking at it. Two different viewing windows showing most of what goes on with these functions are in Figure 6 and Figure 7.

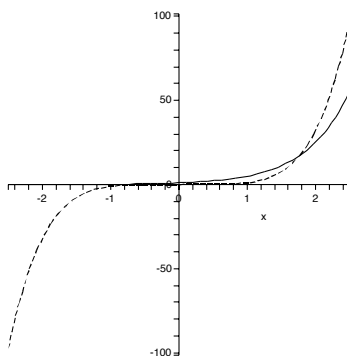


FIGURE 6. Problem 26:  $y = x^5$  (dashed) and  $y = 5^x$  (solid).

29. (a) There are  $100(2^5) = 3200$  bacteria.  
 (b) That would be  $100(2^{t/3})$  bacteria.  
 (c) It would be  $100(2^{20/3})$ . This is between  $100(2^6) = 6400$  and  $100(2^7) = 12,800$ , closer to the latter than to the former. A more precise estimate would be 10159.36673259647663841091609134277286650555729855905926277302887568.  
 (d) The plot is shown in Figure 8. It would appear that the population reaches 50,000 after not quite 27 hours. Armed with the log function from the next

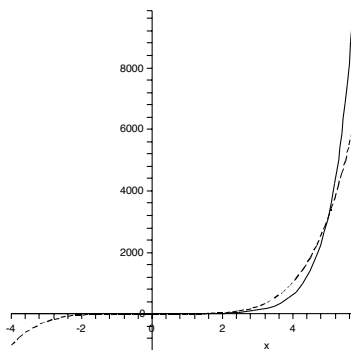


FIGURE 7. Problem 26:  $y = x^5$  (dashed) and  $y = 5^x$  (solid).

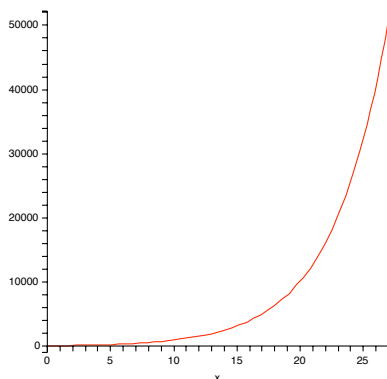


FIGURE 8. Problem 29:  $y = 100(2^{t/3})$ .

section, one can say that the population is 50,000 when

$$100(2^{x/3}) = 50,000$$

$$2^{x/3} = 500$$

$$x = 3 \log_2 500 = 3 \left( \frac{\ln 500}{\ln 2} \right) \doteq 26.8973528539862611308328748.$$

#### SECTION 1.6

3.  $f$  is not one-to-one because  $f(2) = f(6)$ .
5. This function is obviously not one-to-one, since many horizontal lines hit the graph twice.
9.  $f$  is not one-to-one. One could see this by plotting the function and observing that it fails the horizontal line test, which is easy enough to do with a parabola. One

could also work algebraically: pick a  $y$ , say  $y = 0$ . We can then solve

$$\begin{aligned} f(x) &= 0 \\ x^2 - 2x &= 0 \\ x(x - 2) &= 0. \end{aligned}$$

This shows that  $f(0) = f(2) = 0$ ; so  $f$  is not one-to-one.

**25.** The usual approach would be to swap  $x$  and  $y$  and then solve for  $y$ :

$$\begin{aligned} x &= \ln(y + 3) \\ e^x &= e^{\ln(y+3)} \\ e^x &= y + 3 \\ e^x - 3 &= y. \end{aligned}$$

Thus,  $f^{-1}(x) = e^x - 3$ .

**28.** The inverse of the function  $f(x) = 2 - e^x$  should be given by

$$\begin{aligned} x &= 2 - e^y \\ e^y &= 2 - x \\ y &= \ln(2 - x). \end{aligned}$$

Thus,

$$f^{-1}(x) = \ln(2 - x)$$

In Figure 9, I plot  $f$  and  $f^{-1}$ . It's worth taking a moment to think a bit about these plots. In particular, where does each graph have an asymptote?

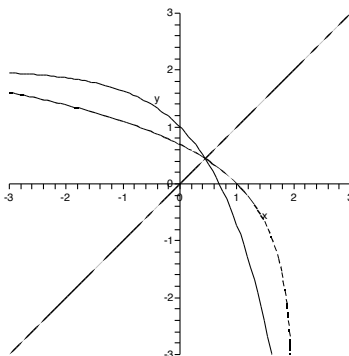


FIGURE 9. Problem 28:  $f(x) = 2 - e^x$  and its inverse.

**35. (b)**  $\log_3(1/27) = -3$ , since  $1/27 = 3^{-3}$ .

**36. (b)**  $\log_{10}(\sqrt{10}) = \frac{1}{2}$ , since  $\sqrt{10} = 10^{1/2}$ .

**37. (b)**  $\log_3 100 - \log_3 18 - \log_3 50 = \log_3 \left( \frac{100}{18 \cdot 50} \right) = \log_3(1/9) = -2$ , since  $1/9 = 3^{-2}$ .

38. (b)  $\ln(\ln(e^{e^{10}})) = \ln(e^{10}) = 10$ .

46. This problem is sort of the dual of Problem 20 in Section 1.5. There we saw evidence that any exponential function grows faster than any power of  $x$ . Here, we see evidence that any logarithm grows slower than any power of  $x$ . There are two points of intersection. *Maple* easily finds the first one near  $x = 3.059726680$ . The behavior of the two functions near this point is shown in Figure 10.

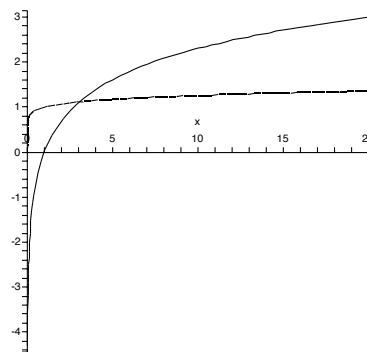


FIGURE 10. Problem 46:  $y = \ln x$  (solid) and  $y = x^{0.1}$  (dotted).

To find the other point of intersection, one can ask *Sage*'s `find_root` to find a root between 5 and  $\infty$ . *Sage* then delivers up the answer that  $x$  is around  $3.430631121 \times 10^{15}$ . Alternatively, one could have tried out various large values of  $x$ : 1,000,000,  $10^{100}$ ,  $10^{1,000,000}$ , etc., trying to find where the crossing point was.

A viewing window showing the second crossing point is shown in Figure 11.

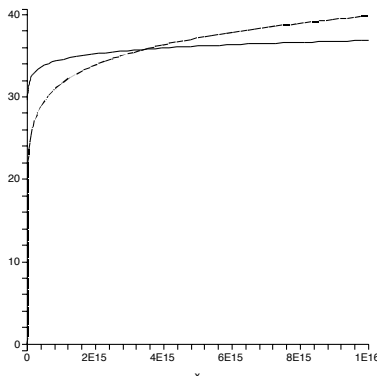


FIGURE 11. Problem 46:  $y = \ln x$  (solid) and  $y = x^{0.1}$  (dotted).

49. (a)  $e^{7-4x} = 6 \iff 7 - 4x = \ln 6 \iff x = \frac{1}{4}(7 - \ln 6)$ .

50. (a)  $\ln(x^2 - 1) = 3 \iff x^2 - 1 = e^3 \iff x = \pm\sqrt{e^3 + 1}$ .

51. (a)  $2^{x-5} = 3 \iff x - 5 = \log_2 3 \iff x = 5 + \log_2 3$ .

52. (a)  $\ln(\ln x) = 1 \iff \ln x = e^1 = e \iff x = e^e$ .

56. You can only take logs of positive numbers, so in order for  $x$  to be in the domain, one must have  $2 + \ln x > 0$ ; that is, one must have  $x > e^{-2}$ .

To get the inverse function, interchange  $x$  and  $y$  and solve for  $y$ :

$$x = \ln(2 + \ln y)$$

$$e^x = 2 + \ln y$$

$$e^x - 2 = \ln y$$

$$e^{e^x - 2} = y.$$

The inverse function is therefore  $f^{-1}(x) = e^{e^x - 2}$ . It's worth plotting both  $f$  and  $f^{-1}$  in order to get some picture of how these functions look.