CALCULUS A
HOMEWORK 4 SOLUTIONS

Section 2.4

4. The function f is continuous on the intervals $[-4,-2)$, $(-2,2)$, $(2,4)$, $(4,6)$, and $(6,8)$.

18. Except at $x=3$, the $f(x)$ is defined and factors as

$$f(x) = \frac{2x^2-5x-3}{x-3} = \frac{(x-3)(2x+1)}{x-3} = 2x+1.$$

Therefore

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}(2x+1) = 7.$$

On the other hand, $f(3) = 6 \neq 7$, so the limit is not equal to the value, and the function is discontinuous at $x=3$.

The graph of f is the graph of the straight line $y = 2x+1$ with a hole at the point $(3,7)$ and with a single point added at $(3,6)$.

26. This is nicely subtle. The tangent function has vertical asymptotes at every odd multiple of $\pi/2$ (that is, at $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots$). Our function is therefore discontinuous at all these points, at least on the ground that it is undefined there. Further, at every even multiple of $\pi/2$ (that is, at $0, \pi, 2\pi, \dots$), $\tan^2 x = 0$; $\ln\tan^2 x$ has a vertical asymptote at these points. Again, our function is undefined at these points, and therefore discontinuous there. At every other point, the function is continuous by an argument like the one in Problem 19.

The graph, in Figure 1, shows what we could work out on our own with a little thought: that all the points of discontinuity are vertical asymptotes.

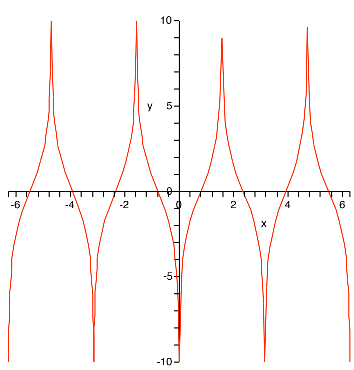

Figure 1. Problem 2.4.26: $y = \ln(\tan^2 x)$.

46. To use the IVT, look for a solution to $\ln x = 3 - 2x$, i.e., to $(\ln x) - 3 + 2x = 0$. The function $f(x) = (\ln x) - 3 + 2x$ is a sum of a logarithmic function and a polynomial, both of which are continuous for all $x > 0$ by Theorem 7. Theorem 4 therefore guarantees that $f(x)$ is everywhere continuous. Clearly

$$f(1) = -1 < 0 < (\ln 2) + 1 = f(2).$$

The IVT therefore assures us of the existence of a point between 1 and 2 at which $f(x) = (\ln x) - 3 + 2x = 0$. To find this point, just keep cutting down the interval in which the sign change must occur. For instance, $f(1.5) = 0.405\ldots$; so there must be a root between 1 and 1.5. Continuing on this way, we find that

$$f(1.34) = -0.0273\ldots < 0 < 0.000105\ldots = f(1.35).$$

An interval of length 0.01 in which a root is guaranteed to exist is therefore $(1.34, 1.35)$.

In passing, *Sage's* **find_root** makes quick work of finding the root, locating it at $x = 1.3499618380355236$. *Sage* can't solve the equation exactly, but some other computer algebra systems can. *Maple* writes the solution using a special function you are unlikely to have met: the Lambert W-function. In terms of this function, the root is writes the root as $x = e^{-\mathrm{LambertW}(2e^3)} + 3$.

Section 2.5

2. (a) The graph of $y = f(x)$ can in fact cross either a vertical asymptote or a horizontal asymptote. All that is required for there to be a vertical asymptote is that the limit (one-sided or two-sided) as $x \to a$ be $\pm\infty$. All that is required for there to be a horizontal asymptote is that the limit as $x \to \pm\infty$ exists. Figures 2,3 show graphs of functions which intersect their vertical and horizontal asymptotes, respectively.

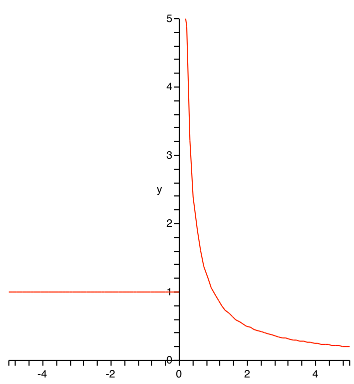

Figure 2. A function which intersects its vertical asymptote

(b) A function can have 0, 1, or 2 horizontal asymptotes. Figure 3 shows a function with one horizontal asymptote; Figures 4,5 show functions with 0 or 2 horizontal asymptotes, respectively.

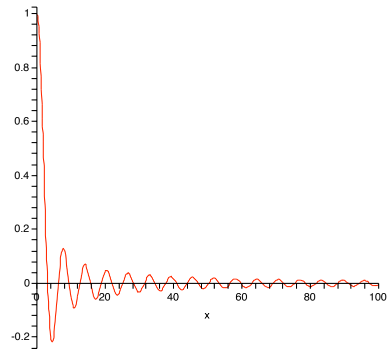

FIGURE 3. A function which intersects its (one) horizontal asymptote

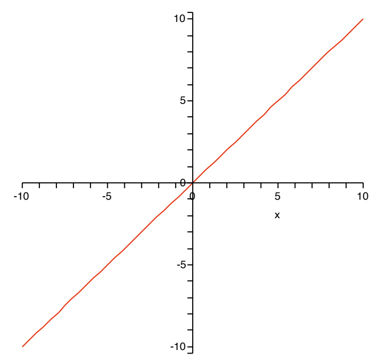

FIGURE 4. A function with no horizontal asymptotes

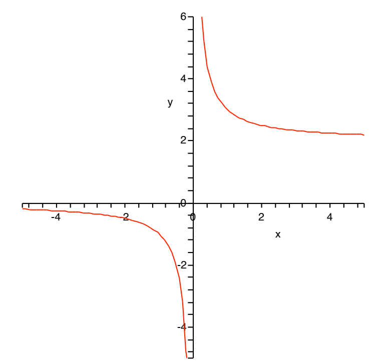

FIGURE 5. A function with two horizontal asymptotes

x	$f(x)$
0.9	-3.32
0.99	-33.33
0.999	-333.33
0.9999	-3333.33
1.1	3.32
1.01	33.33
1.001	333.33
1.00001	3333.33

TABLE 1. Problem 2.5.12: Values of $f(x) = \dfrac{1}{x^3-1}$ near $x = 1$.

4. (a) $\lim\limits_{x\to\infty} g(x) = 2$.

(b) $\lim\limits_{x\to-\infty} g(x) = -2$.

(c) $\lim\limits_{x\to 3} g(x) = \infty$.

(d) $\lim\limits_{x\to 0} g(x) = -\infty$.

(e) $\lim\limits_{x\to -2^+} g(x) = -\infty$.

(f) The equations of the asymptotes are: $y = 2$ and $y = -2$ (the two horizontal asymptotes), and $x = -2$, $x = 0$, and $x = 3$ (the three vertical asymptotes).

12. (a) Table 1 shows $f(x) = \frac{1}{x^3-1}$ for various x close to $x = 1$.

It appears that $\lim\limits_{x\to 1^-} f(x) = -\infty$, and $\lim\limits_{x\to 1^+} f(x) = \infty$.

(b) If x is close to 1 and a little larger, then the denominator for $f(x)$ will be a positive number that's very close to 0, while the numerator is (of course) 1. As x gets close to 1, the denominator will get closer to 0 but remain positive. The limit from the right is therefore ∞.

On the other hand, if x is close to 1 and a little smaller, then the denominator for $f(x)$ will be a negative number that's very close to 0, and the numerator is still 1. As x gets close to 1, the denominator will get closer to 0 but remain negative. The limit from the left is therefore $-\infty$.

(c) Figure 6 shows a graph of $f(x)$, and indeed it supports our claim that the one-sided limits as $x \to 1$ are ∞ and $-\infty$, respectively.

15. Either just to the right or just to the left of $x = 1$, the numerator of $\frac{2-x}{(x-1)^2}$ is close to 1 and the denominator is tiny and positive. On either side of 1, the function therefore approaches $+\infty$:

$$\lim_{x\to 1} \frac{2-x}{(x-1)^2} = +\infty.$$

16. If x is just to the left of -3, then $x+2$ is close to -1 and $x+3$ is tiny and negative. The quotient of the two is therefore large and positive:

$$\lim_{x\to -3^-} \frac{x+2}{x+3} = +\infty.$$

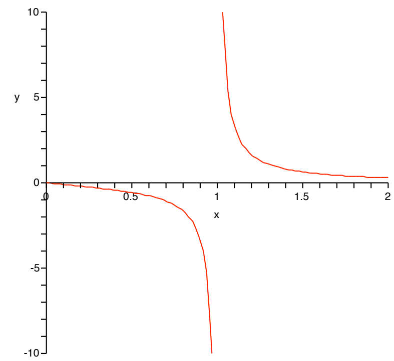

FIGURE 6. $y = \frac{1}{x^3-1}$ near $x = 1$.

22.

$$\begin{aligned}\lim_{x\to\infty} \frac{3x+5}{x-4} &= \lim_{x\to\infty} \frac{3+\frac{5}{x}}{1-\frac{4}{x}} \\ &= \lim_{x\to\infty} \frac{3}{1} = 3.\end{aligned}$$

24.

$$\begin{aligned}\lim_{t\to-\infty} \frac{t^2+2}{t^3+t^2-1} &= \lim_{t\to-\infty} \frac{\frac{1}{t}+\frac{2}{t^3}}{1+\frac{1}{t}-\frac{1}{t^3}} \\ &= \lim_{t\to-\infty} \frac{0}{1} = 0.\end{aligned}$$

39. First let's look for horizontal asymptotes by looking at the limits as $x \to \pm\infty$.

$$\begin{aligned}\lim_{x\to\infty} \frac{2x^2+x-1}{x^2+x-2} &= \lim_{x\to\infty} \frac{2+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}} \\ &= \lim_{x\to\infty} \frac{2}{1} = 2.\end{aligned}$$

A similar calculation will show that $\lim_{x\to-\infty}$ is also 2. So we have a horizontal asymptote at $y = 2$, and it's an asymptote in both the positive and negative directions.

Now let's look for vertical asymptotes. There's a possibility for a vertical asymptote anywhere there's a 0 in the denominator. Since the denominator factors as $x^2 + x - 2 = (x+2)(x-1)$, we should look for a possible vertical asymptote at $x = -2$ and at $x = 1$.

To find the one-sided limits as $x \to -2$, note that when x is just larger than -2, the numerator is close to $2(-2)^2 - 2 - 1 = 5$, while the denominator is close to 0, and negative (since $x + 2$ will be positive, but $x - 1$ will be negative). So as $x \to -2^+$, $y \to -\infty$. On the other hand, when x is just smaller than -2, the numerator is still close to 5, and the denominator is still close to 0, but now the denominator is positive. So as $x \to -2^-$, $y \to \infty$.

To find the one-sided limits as $x \to 1$, note that when x is just larger than 1, the numerator is close to $2(1)^2 + 1 - 1 = 2$, while the denominator is close to 0, and positive (since $x + 2$ and $x - 1$ will both be positive). So as $x \to 1^+$, $y \to \infty$. On the other hand, when x is just smaller than 1, the numerator is still close to 2, and the denominator is still close to 0, but now the denominator is negative. So as $x \to 1^-$, $y \to -\infty$.

Figure 7 shows the graph of the function, including both the horizontal and the vertical asymptotes.

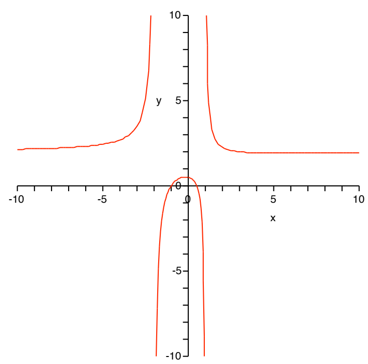

FIGURE 7. $y = \frac{2x^2+x-1}{x^2+x-2}$

SECTION 2.6

4. (a) Definition 1 says that the slope of the tangent line is

$$\lim_{x\to 1} \frac{(x - x^3) - 0}{x - 1} = \lim_{x\to 1} \frac{x(1-x)(1+x)}{x-1} = \lim_{x\to 1} [-x(1+x)] = -2.$$

(b) Equation 2 would have us write the limit as

$$\begin{aligned} \lim_{h\to 0} \frac{[(1+h) - (1+h)^3] - 0}{h} &= \lim_{h\to 0} \frac{[(1+h) - (1 + 3h + 3h^2 + h^3)] - 0}{h} \\ &= \lim_{h\to 0} \frac{-2h - 3h^2 - h^3}{h} \\ &= \lim_{h\to 0} (-2 - 3h - h^2) = -2. \end{aligned}$$

(c) The tangent line is the line of slope -2 through the point $(1, 0)$. In other words, it's the line $y = -2x + 2$. I'll let you do the graphs, but there's one at an intermediate level of zoom in Figure 8.

6. The slope of the tangent line will be

$$\lim_{x\to 2} \frac{(x^3 - 3x + 1) - 3}{x - 2} = \lim_{x\to 2} \frac{x^3 - 3x - 2}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x^2 + 2x + 1)}{x - 2} = \lim_{x\to 2} (x^2 + 2x + 1) = 9.$$

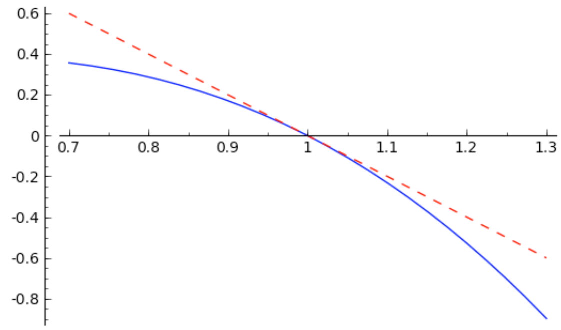

FIGURE 8. Problem 2.6.4: $y = x - x^3$ and its tangent line near $x = 1$.

The tangent line is therefore the line of slope 9 through the point $(2, 3)$. The equation of this line is $y - 3 = 9(x - 2)$, or $y = 9x - 15$.

8. The slope of the tangent line is

$$\lim_{h\to 0} \frac{\frac{2(1+h)+1}{(1+h)+2} - 1}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{2(1+h)+1}{(1+h)+2} - 1\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2h+3}{h+3} - 1\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{h+3}\right] = \lim_{h\to 0} \frac{1}{h+3} = \frac{1}{3}.$$

The tangent line is therefore the line of slope $\frac{1}{3}$ through the point $(1, 1)$. The equation of this line is $y - 1 = \frac{1}{3}(x - 1)$, or $y = \frac{1}{3}x + \frac{2}{3}$.

11. The particle is moving the right when its derivative is positive, i.e., when $0 < x < 1$ or $4 < x < 6$. It is moving to the left when the derivative is negative, i.e., when $2 < x < 3$. It is stationary when $f'(x) = 0$, i.e., when $1 < x < 2$ and when $3 < x < 4$.

I've been careful not to include the points at $x = 0, 1, 2, 3, 4, 6$ in any of these sets because at those points, the derivative is undefined. I therefore think that at those points it would be incorrect to make any of the claims about left, right, or stationary. This is a detail for which I would be reluctant to take off points, though.

The velocity function is sketched by *Sage* in Figure 9. Properly, there should be little open circles at the end of every segment of the graph.

12. **(a)** Runner A runs the 100 m at a constant velocity. Runner B, on the other hand, starts out going slowly and speeds up throughout the race. In the beginning, he's going about 2m/s, a brisk walking pace. By the end of the race, he has speeded up to about 40m/s, which is roughly 90 mph. I'm guessing he failed the blood test.

(b) When the vertical distance between the graphs showing distances they have traveled is as large as possible (i.e., when the two curves have the same slope). This looks to happen around 10 seconds into the race.

(c) At that point about 10 seconds into the race when the slopes are the same.

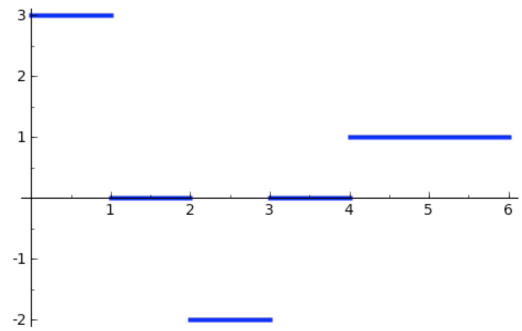

FIGURE 9. Problem 2.6.11: The velocity function.

17. Note that only $g'(0)$ is negative; the other three slopes are positive. The graph of $g(x)$ is steeper at $x = -2$ than at $x = 2$, and steeper at $x = 2$ than at $x = 4$. So the correct order is $g'(0)$, 0, $g'(4)$, $g'(2)$, $g'(-2)$.

18. The tangent line will be the line through $(5, -3)$ with slope 4. This line has equation $y + 3 = 4(x - 5)$, or $y = 4x - 23$.

20. We're talking about a tangent line to $f(x)$ at the point $(4, 3)$, so this must be a point on the graph. Therefore $f(4) = 3$. The line through $(4, 3)$ and $(0, 2)$ has slope $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

22. One such function is in Figure 10.

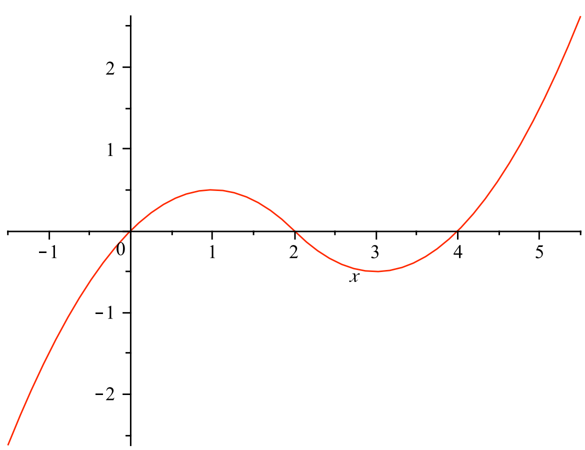

FIGURE 10. Problem 2.6.22: A function with some values and slopes.

24. If $g(x) = x^4 - 2$, then

$$\begin{aligned} g'(1) &= \lim_{x\to 1} \frac{(x^4-2)+1}{x-1} = \lim_{x\to 1} \frac{x^4-1}{x-1} \\ &= \lim_{x\to 1} \frac{(x-1)(x^3+x^2+x+1)}{x-1} = \lim_{x\to 1}(x^3+x^2+x+1) = 4. \end{aligned}$$

The tangent line is therefore the line with slope 4 through the point $(1,-1)$, i.e., $y+1 = 4(x-1)$, or $y = 4x-5$. The function and its tangent line are shown in Figure 11.

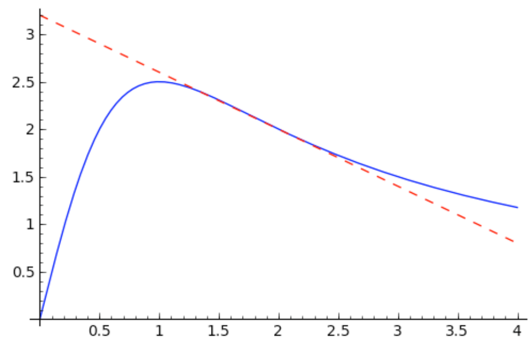

FIGURE 11. $y = \dfrac{5x}{1+x^2}$ and the line tangent to it at $x = 2$.

25. Formula 2 would tell us that

$$\begin{aligned} F'(2) &= \lim_{h\to 0} \frac{\frac{5(2+h)}{1+(2+h)^2} - 2}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{5(2+h)}{1+(2+h)^2} - 2\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{5h+10}{h^2+4h+5} - \frac{2(h^2+4h+5)}{h^2+4h+5}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2h^2-3h}{h^2+4h+5}\right] \\ &= \lim_{h\to 0} \frac{-2h-3}{h^2+4h+5} = -\frac{3}{5}. \end{aligned}$$

The tangent line is therefore the line with slope $-\frac{3}{5}$ through the point $(2,2)$, i.e., $y-2 = -\frac{3}{5}(x-2)$, or $y = -\frac{3}{5}x + \frac{16}{5}$.

28. If $f(t) = 2t^3 + t$, then

$$\begin{aligned} f'(a) &= \lim_{t\to a} \frac{(2t^3+t)-(2a^3+a)}{t-a} = \lim_{t\to a} \frac{2(t^3-a^3)+(t-a)}{t-a} \\ &= \lim_{t\to a} \frac{2(t-a)(t^2+at+a^2)+(t-a)}{t-a} = \lim_{t\to a}[2(t^2+at+a^2)+1] = 6a^2+1 \end{aligned}$$

29. If $f(t) = (2t+1)/(t+3)$, then

$$\begin{aligned} f'(a) &= \lim_{t\to a} \frac{\frac{2t+1}{t+3} - \frac{2a+1}{a+3}}{t-a} \\ &= \lim_{t\to a} \frac{(2t+1)(a+3) - (2a+1)(t+3)}{(t-a)(a+3)(t+3)} \\ &= \lim_{t\to a} \frac{[2at + a + 6t + 3] - [2at + 6a + t + 3]}{(t-a)(a+3)(t+3)} \\ &= \lim_{t\to a} \frac{5t - 5a}{(t-a)(a+3)(t+3)} \\ &= \lim_{t\to a} \frac{5}{(a+3)(t+3)} = \frac{5}{(a+3)^2}. \end{aligned}$$

31. If $f(x) = \sqrt{1-2x}$, then

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{\sqrt{1-2x} - \sqrt{1-2a}}{x-a} = \lim_{x\to a} \frac{\sqrt{1-2x} - \sqrt{1-2a}}{x-a} \cdot \frac{\sqrt{1-2x} + \sqrt{1-2a}}{\sqrt{1-2x} + \sqrt{1-2a}} \\ &= \lim_{x\to a} \frac{(1-2x) - (1-2a)}{(x-a)(\sqrt{1-2x} + \sqrt{1-2a})} = \lim_{x\to a} \frac{-2}{\sqrt{1-2x} + \sqrt{1-2a}} = -\frac{1}{\sqrt{1-2a}}. \end{aligned}$$

36. This is the derivative of $f(x) = \tan x$ at $a = \frac{\pi}{4}$.

44. **(a)** The average rate of growth from 2005 to 2007 is 2385 locations per year. From 2005 to 2006, it's 2199 locations per year. From 2004 to 2005, it's 1672 locations per year.

(b) Presumably we should average the last two intervals in part (a) to get 1935.5 locations per year. You might observe that this is also the average rate of increase from 2004 to 2006. Can you work out algebraically why this will always be the case?

(c) I guess the intention here is for you to sketch the curve and then try to eyeball in the tangent line. That's actually a bit tricky, since if you just connect the dots you get a polygonal line with an angle at $t = 2005$, and therefore without a tangent line there. The angular curve and the calculated "tangent" line are shown in Figure 12.

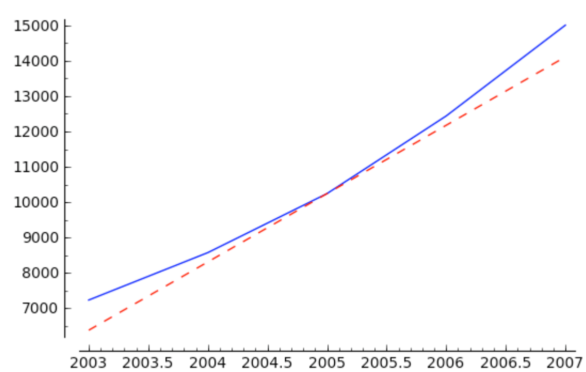

FIGURE 12. Problem 2.6.44: Starbucks outlets as a function of time.

47. **(a)** The derivative $f'(x)$ is the rate of change in the cost of the gold as a function of the number of ounces produced. Roughly, it's the cost in dollars of each additional ounce produced. The units are dollars/ounce.

(b) It means that the tangent to the curve at $x = 800$ has slope 17 dollars/ounce. As long as the curve itself lies close to the tangent line, each additional ounce of gold produced will therefore cost an extra \$17. For that cost, one should definitely increase production.

(c) In the long run, as you mine out the high quality ore, the cost of producing an additional ounce of gold will rise. In the short term, it's hard to know. Stewart seems to think it will decrease, presumably because he's imagining that the mine is so new that you're still paying initial capital costs, sinking the first hole, and so on. I'm not so sure. If you said, "decrease," you ought to have given an explanation.