CALCULUS A
HOMEWORK 6 SOLUTIONS

Section 3.1

2. **(a)** A graph of $y = e^x$ is shown in Figure 1. Note in particular that the graph crosses the y-axis with a slope of 1. We know this slope is 1 because the *definition* of e is that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$.

(b) $f(x) = e^x$ is an exponential function, while $g(x) = x^e$ is a power function. The derivatives are calculated using the formulas for these two different types of functions: $f'(x) = e^x$, and $g'(x) = ex^{e-1}$.

(c) When x gets large, the exponential function $f(x) = e^x$ grows more rapidly than the power function $g(x)$, as Figure 2 shows.

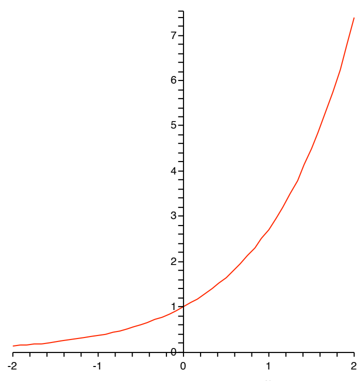

Figure 1. $y = e^x$

4. $f'(x) = 0$: $\sqrt{30}$ is a constant.

8. $f'(t) = 3t^5 - 12t^3 + 1$.

15. $y = 3e^x + 4x^{-1/3}$; so $dy/dx = 3e^x - \frac{4}{3}x^{-4/3} = 3e^x - \dfrac{4}{3x\sqrt[3]{x}}$.

17. This will be easy once we have the Chain Rule, but even now, we can say that $F(x) = \frac{1}{32}x^5$ and that $F'(x) = \frac{5}{32}x^4$.

18. This appears at first blush to require the Quotient Rule, which we don't yet know; but a moment's reflection shows that $f(x) = 1 - 3x^{-1} + x^{-2}$, so that

$$f'(x) = 3x^{-2} - 2x^{-3} = \frac{3}{x^2} - \frac{2}{x^3} = \frac{3x-2}{x^2}.$$

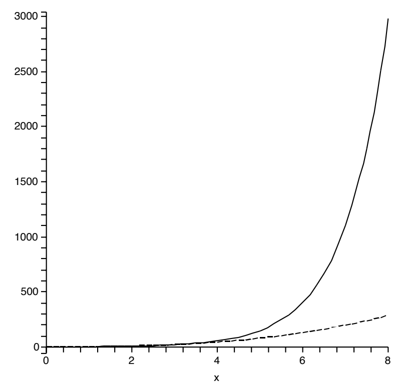

FIGURE 2. $f(x) = e^x$ (solid) and $g(x) = x^e$ (dashed)

22. $y = ae^v + bv^{-1} + cv^{-2}$, so $y' = ae^v + b(-1v^{-2}) + c(-2v^{-3)} = ae^v - \frac{b}{v^2} - \frac{2c}{v^3}$.

36. $f'(x) = 1 + (-1x^{-2}) = 1 - \frac{1}{x^2}$. Figure 3 shows a graph of f and f'. Note that f is concave up for x positive and concave down for x negative, and correspondingly f' is increasing for x positive and decreasing for x negative. In addition, f is increasing on the intervals $(\infty, -1)$ and $(1, \infty)$, where f' is positive; f is decreasing on the intervals $(-1, 0)$ and $(0, 1)$, where f' is negative. Both functions have a vertical asymptote at $x = 0$. It makes sense that f' has an asymptote there because the slope of the tangent line to f as $x \to 0$ (from either side) approaches $-\infty$.

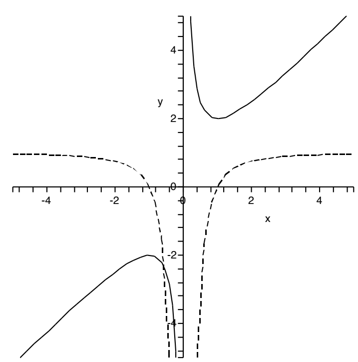

FIGURE 3. Problem 36: $f(x) = x + \frac{1}{x}$ (solid) and $f'(x) = 1 - \frac{1}{x^2}$ (dashed)

38. The exact value is, of course,

$$-\frac{1}{2x^{3/2}}\bigg|_{x=4} = -\frac{1}{16}.$$

Your numerical estimates should be close to this.

44. $f'(x) = e^x - 3x^2$, and $f''(x) = e^x - 6x$. Figure 4 shows a graph of f, f', and f''. Note that f'' is positive until $x \approx 0.2$, then negative until $x \approx 2.8$, then positive again; correspondingly, f' is increasing until $x \approx 0.2$, then decreasing until $x \approx 2.8$, then increasing again; correspondingly, f is concave up until $x \approx 0.2$, then concave down until $x \approx 2.8$, then concave up again. Also, f' is negative until $x \approx -0.4$, then positive until $x \approx 0.9$, then negative again until $x \approx 3.7$, then positive again; correspondingly, f is decreasing until $x \approx -0.4$, then increasing until $x \approx 0.9$, then decreasing again until $x \approx 3.7$, then increasing again.

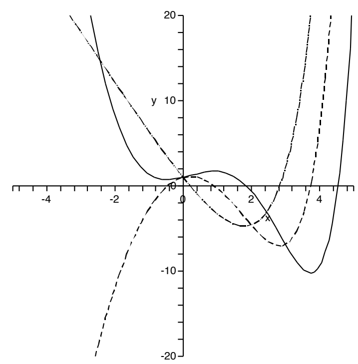

FIGURE 4. Problem 44: f (solid), f' (dashed), and f'' (dotted)

47. The function $f(x) = 5x - e^x$ is increasing where $f'(x) = 5 - e^x \geq 0$, i.e., where $x \leq \ln 5$.

50. The tangent to $f(x)$ will be horizontal whenever $f'(x) = 0$. By calculation, $f'(x) = 3x^2 + 6x + 1$. This is zero when

$$x = \frac{-6 \pm \sqrt{24}}{6} = -1 \pm \tfrac{1}{3}\sqrt{6}.$$

71. Note that the limit is the slope of the tangent line to $f(x) = x^{1000}$ at $x = 1$. $f'(x) = 1000x^{999}$, so $f'(1) = 1000$. Therefore the limit is 1000.

SECTION 3.2

1. The Product Rule would give

$$4x(x - x^2) + (1 + 2x^2)(1 - 2x) = -8x^3 + 6x^2 - 2x + 1.$$

When you multiply it out, $f(x) = -2x^4 + 2x^3 - x^2 + x$. The derivative of this is just what the Product Rule gives.

2. The Quotient Rule would give us

$$F'(x) = \frac{x^2\left(4x^3 - 15x^2 + \frac{1}{2\sqrt{x}}\right) - \left(x^4 - 5x^3 + \sqrt{x}\right)2x}{x^4}$$
$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \frac{3}{2}x^{-5/2}.$$

Alternatively, we could first do the division and write

$$F(x) = x^2 - 5x + x^{-3/2},$$

whose derivative is obviously what the Quotient Rule gave us. Clearly the second approach is better.

I said in class that if a problem just asks you to compute a derivative, you don't need to simplify the result. That's true—you should get full credit for just using the differentiation rules. That said, in solving the following problems I have sometimes done some simplification if a small amount of simplification made a big difference in the result. When to simplify and when not is always a complicated judgment call, but if a little work makes a big difference, then why not take advantage of it?

4. $g'(x) = \sqrt{x}\,e^x + \dfrac{1}{2\sqrt{x}}\,e^x.$

6. $y' = \dfrac{(1+x)e^x - e^x \cdot 1}{(1+x)^2}.$

10. $R'(t) = (t + e^t)\left(-\dfrac{1}{2\sqrt{t}}\right) + (1+e^t)(3-\sqrt{t}).$

12. $y' = \dfrac{(x^3+x-2)1 - (x+1)(3x^2+1)}{(x^3+x-2)^2} = \dfrac{-2\,x^3+3\,x^2+3}{(x-1)^2(x^2+x+2)^2}.$

17. I would do this one by first simplifying y and then differentiating:

$$y = v^2 - 2\sqrt{v}$$
$$y' = 2v - \frac{1}{\sqrt{v}}.$$

Of course, you could instead have used the Quotient Rule and gotten

$$\frac{v(3v^2 - 3\sqrt{v}) - (v^3 - 2v\sqrt{v})1}{v^2},$$

or even more horrible things if you used the Product Rule to handle $2v\sqrt{v}$. Some algebra will convince you thaat these results are the same; but it is fairly obvious you don't want to do it this way.

23. Again, there are at least 2 possibilities. One could just use the Quotient Rule without any preliminaries, and get

$$\frac{\left(x + \frac{c}{x}\right)1 - x\left(1 - \frac{c}{x^2}\right)}{\left(x + \frac{c}{x}\right)^2}.$$

Alternatively, one could start out by multiplying top and bottom by x, and then using the Quotient Rule. This gives yiou less of a mess to clean up:

$$f(x) = \frac{x^2}{x^2+c}$$
$$f'(x) = \frac{(x^2+c)2x - x^2 \cdot 2x}{(x^2+c)^2} = \frac{2cx}{(x^2+c)^2}.$$

As before, advance clean-up pays off in the end.

33. **(a)** The Quotient Rule (or the Reciprocal Rule) tells us that

$$y' = -\frac{2x}{(1+x^2)^2}.$$

This means that $y'(-1) = \frac{1}{2}$. The tangent line at $(-1, \frac{1}{2})$ therefore has equation $y - \frac{1}{2} = \frac{1}{2}(x+1)$, or $y = \frac{1}{2}x + 1$.

(b) See Figure 5.

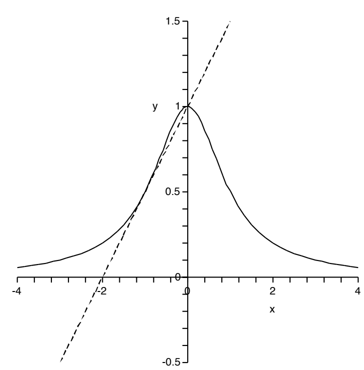

FIGURE 5. Prob. 3.2.33: The witch of Agnesi and its tangent.

44. The Quotient Rule tells us that

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2} = \frac{xh'(x) - h(x)1}{x^2}\bigg|_{x=2} = \frac{2h'(2) - h(2)}{2^2} = \frac{2(-3) - 4}{4} = -\frac{5}{2}.$$

45. **(a)** $u'(1) = f(1)g'(1) + f'(1)g(1) = 2(-1) + 1(2) = 0.$

(b) The Quotient Rule gives

$$v'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{g(5)^2} = \frac{2\left(-\frac{1}{3}\right) - 3\left(\frac{2}{3}\right)}{2^2} = -\frac{2}{3}.$$

49. Let $p(t)$ be the population at time t, $i(t)$ be the per capita income at time t, and $g(t)$ be the gross income of the region. Then $g(t) = p(t)\,i(t)$; so

$$g'(t) = p(t)i'(t) + i(t)p'(t). \tag{1}$$

In our case, that means

$$g'(t) = 961,400 \cdot 1400 + 30,953 \cdot 9200 = \$1,630,727,600/\text{year}.$$

(Think about why these are the right units.)

The first term on the right hand side of equation (1) represents the rate at which income increases because existing individuals are gaining income; the second term represents the rate at which income increases because new residents arrive with new incomes.

51. If $f(x) = x^3e^x$, then $f'(x) = 3x^2e^x + x^3e^x = (x^3 + 3x^2)e^x = x^2(x+3)e^x$. To determine where $f'(x)$ is positive and where it is negative is easy, since $x^2 \geq 0$ and $e^x > 0$ for all x. The sign of $f'(x)$ therefore depends only on that of $x + 3$: $f'(x) < 0$ if $x < -3$, and $f'(x) \geq 0$ if $x \geq -3$. Thus, f is increasing on $[-3, \infty)$ and decreasing on $(-\infty, -3]$. You might enjoy convincing yourselves that I have not lied about the behavior of f at $x = 0$.

Section 3.3

2. $y' = -2 \csc x \, \cot x - 5 \sin x$.

4. $f'(x) = \sqrt{x} \, \cos x - \dfrac{\sin x}{2\sqrt{x}}$.

5. $y' = (\sec\theta \, \tan\theta)(\tan\theta) + (\sec\theta)(\sec^2\theta)$.

10. $y' = \dfrac{(x + \cos x)\cos x - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2} = \dfrac{x \, \cos x}{(x + \cos x)^2}$.

16. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x} = \sec x \, \tan x$.

23. The slope of the tangent line will be

$$[2 \sin x + 2x \, \cos x]_{x=\pi/2} = 2.$$

The tangent line will go through the point $(\pi/2, \; \pi)$. The equation of the tangent line will therefore be $y - \pi = 2(x - \pi/2)$, i.e., $y = 2x$. The curve and its tangent line are shown in Figure 6. Notice that the tangent line seems to be tangent to the curve at more than one point. It might be interesting to try to prove this fact.

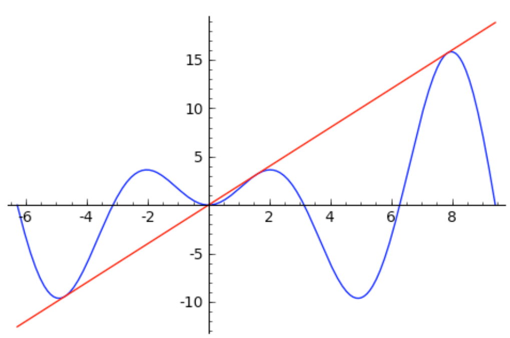

Figure 6. The curve $y = 2\sin x + 2x \, \cos x$ and one of its tangent lines.

31. The graph of $f(x)$ has a horizontal tangent whenever the derivative is 0. $f'(x) = 1 + 2\cos x$, so $f'(x) = 0$ whenever $1 + 2\cos x = 0$, or whenever $\cos x = -\frac{1}{2}$. This happens for $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$, as well as all multiples of 2π plus each of these.

The graph of $f(x)$ in Figure 7 confirms our calculation.

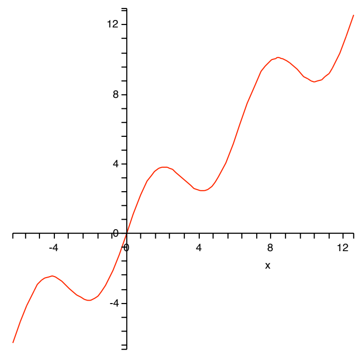

FIGURE 7. Problem 3.3.31: $f(x) = x + 2\sin x$ on $[-2\pi, 4\pi]$

39. The first few derivatives of $f(x) = \sin x$ are:

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f''''(x) &= \sin x \\ f^{(5)}(x) &= \cos x \end{aligned}$$

Notice that $f''''(x) = f(x)$, so the pattern will actually repeat with every fourth derivative. Since 100 is a multiple of 4, we have $f^{(100)}(x) = \sin x$, and the one before it must be $f^{(99)}(x) = -\cos x$.

40. The first few derivatives of $f(x) = x\sin x$ are:

$$\begin{aligned} f'(x) &= \sin x + x\cos x \\ f''(x) &= \cos x + \cos x - x\sin x = 2\cos x - x\sin x \\ f'''(x) &= -2\sin x - \sin x - x\cos x = -3\sin x - x\cos x \\ f''''(x) &= -3\cos x - \cos x + x\sin x = -4\cos x + x\sin x \\ f^{(5)}(x) &= 4\sin x + \sin x + x\cos x = 5\sin x + x\cos x \end{aligned}$$

Again, the pattern repeats every fourth derivative, with the coefficient of the first term increasing by one each time. So since 36 is a multiple of 4, $f^{(35)}(x) = -35\sin x - x\cos x$.

42. (a) With the substitution $\theta = 5x$, $x = \frac{\theta}{5}$. And as $x \to 0$, $\theta \to 0$ as well. So:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 5x}{x} &= \lim_{\frac{\theta}{5}\to 0}\frac{\sin\theta}{\frac{\theta}{5}}\\ &= \lim_{\theta\to 0} 5\frac{\sin\theta}{\theta}\\ &= 5\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 5\end{aligned}$$

(b) Note that we haven't learned the chain rule yet, so no fair using the chain rule to calculate this derivative!

$$\begin{aligned}\frac{d}{dx}(\sin 5x) &= \lim_{h\to 0}\frac{\sin(5x+5h)-\sin 5x}{h}\\ &= \lim_{h\to 0}\frac{\sin 5x\cos 5h + \sin 5h\cos 5x - \sin 5x}{h}\\ &= \lim_{h\to 0}\left(\sin 5x\frac{\cos 5h - 1}{h} + \cos 5x\frac{\sin 5h}{h}\right)\\ &= \sin 5x\lim_{h\to 0}\frac{\cos 5h-1}{h} + \cos 5x\lim_{h\to 0}\frac{\sin 5h}{h}\\ &= \sin 5x\lim_{h\to 0}\frac{\cos 5h-1}{h} + (\cos 5x)5\end{aligned}$$

Note that one of the limits we get is the one we evaluated in part a. The other, we still need to evaluate. As in part a, we use the trick of setting $\theta = 5h$, so that $h = \frac{\theta}{5}$.

$$\begin{aligned}\lim_{h\to 0}\frac{\cos 5h-1}{h} &= \lim_{\frac{\theta}{5}\to 0}\frac{\cos\theta - 1}{\frac{\theta}{5}}\\ &= \lim_{\theta\to 0} 5\frac{\cos\theta-1}{\theta}\\ &= 5\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}\\ &= 5(0) = 0.\end{aligned}$$

Therefore, $\frac{d}{dx}(\sin 5x) = (\sin 5x)0 + (\cos 5x)5 = 5\cos 5x$. This is, of course, the same answer we'd get by using the chain rule.

46. (a) Try calling $u = \frac{1}{x}$, and so $x = \frac{1}{u}$. As $x \to \infty$, $\frac{1}{x} \to 0$, so $u \to 0$.

$$\begin{aligned}\lim_{x\to\infty} x\sin(\frac{1}{x}) &= \lim_{u\to 0^+} \frac{1}{u}\sin u \\ &= \lim_{u\to 0^+} \frac{\sin u}{u} \\ &= 1.\end{aligned}$$

(b) This limit is an excellent candidate for the squeeze theorem. Observe that because $-1 \le \sin\frac{1}{x} \le 1$ for all x, we have $-x \le x\sin\frac{1}{x} \le x$ for all x. As $x \to 0$, both $-x$ and $x \to 0$. Therefore, by the squeeze theorem, $x\sin\frac{1}{x} \to 0$ also.

(c) The graph of $f(x) = x\sin\frac{1}{x}$ is shown in Figure 8, with a closer look in Figure 9, which also shows the squeezing functions $y = x$ and $y = -x$.

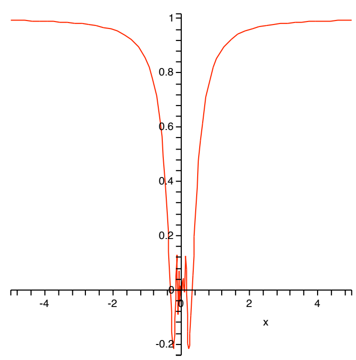

FIGURE 8. Problem 3.3.46: $f(x) = x\sin\frac{1}{x}$

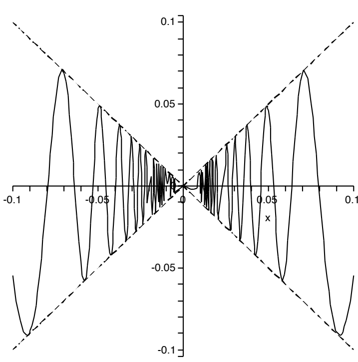

FIGURE 9. Problem 3.3.46: A closer look at $f(x) = x\sin\frac{1}{x}$ near $x = 0$.