CALCULUS A, LAB 2

I'd like to do two things in this lab: first, to take a bit of time exploring limits, and second, to play a bit with mathematics in the Greek style, asking ourselves the question how we would go about computing if we were Archimedes (or maybe, Archimedes equipped with an early version of *Sage* that had everything except π). Feel free to work in pairs unless you're opposed to the idea. For the more conjectural parts of this lab, especially some of the geometry at the end, it may be helpful if pairs consult with other pairs. I'd like each pair to turn in a paper, but if working in larger groups increases your efficiency, then do so. Feel free to wander out to sit together and talk, coming back in here when and if you need computer tools.

Limits

First, I'd like us to look at some limits using both *Sage's* graphing capabilities and its numerical power.

1. Investigate the limit

$$\lim_{x\to 2} \frac{3x^2 - 12}{x - 2}$$

(a) by plotting the function near $x = 2$ and seeing what number, if any, $f(x)$ is getting close to.

(b) by computing the value of $f(x)$ at some points close to $x = 2$. A good initial choice might be 1.9, 1.99, 1.999, 2.001, 2.01, 2.1. Are there patterns in the values you observe which let you infer what the limit might be?

(c) Do you find the analysis in part (a) or the analysis in part (b) more persuasive? Why?

2. Repeat this problem with the limits

$$\lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1}$$

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}$$

3. Plot the pairs of functions

(a) $3x^2 - 12$ and $12(x - 2)$ near the point $x = 2$.

(b) $\sqrt{x} - 1$ and $\frac{1}{2}(x - 1)$ near the point $x = 1$.

(c) $1 - \cos x$ and $\frac{1}{2}x^2$ near the point $x = 0$.

(You want to plot each pair of functions together on a single graph, so that you'll end up with 3 graphs, one for each part of the problem.) How do the limit calculations in Problems 1–2 help you understand these plots?

Archimedes and π

One of the things Archimedes is known for is developing a method for approximating π as accurately as one likes. He ends up giving the approximation that is between $3\frac{1}{7} \doteq 3.142857$ and $3\frac{10}{71} \doteq 3.140845$, though at least with machines to do the arithmetic, one can use his method to get better approximations still. In this series of problems, we won't explore exactly how Archimedes in fact computed π. Instead, we'll use a geometrical approach which is in keeping with what he might have done, and which is probably easier for modern folks to follow. Our starting point is to remember that π is the ratio of the circumference of a circle to its diameter. In particular, a circle with radius 1 will have circumference 2π. We will try to estimate the circumference of a circle of radius 1 by computing the perimeters of polygons inside the circle. By increasing the number of sides of these polygons, we will get better and better approximations to the circumference 2π. Archimedes did something similar, but he used areas instead of lengths, relying on the fact that the area of a circle of radius 1 is π. He also worked both with inscribed and with circumscribed polygons so as to get both lower and upper bounds for π.

4. In a circle of radius 1, inscribe a square as in Figure 1.

(a) Argue that the circumference of the circle is larger than the perimeter of the square.

(b) Compute the lengths of the segment a_4 joining the midpoint of the edge of the square to the center of the circle, and the segment b_4 forming half the edge of the square. Triangles and the Pythagorean Theorem may be useful here.

(c) Compute the perimeter of the square. You know that this perimeter must be less than the circumference 2π of the circle. What estimate does this give you for π?

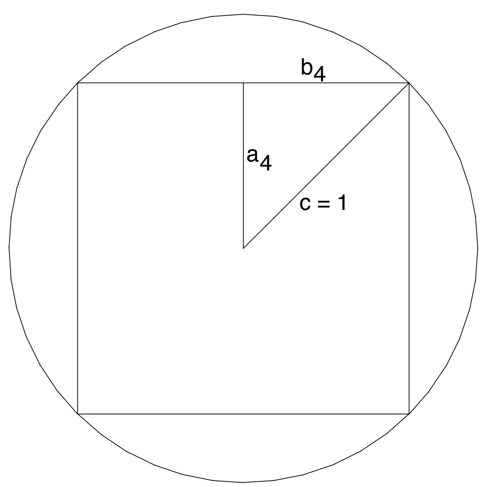

Figure 1. A square inscribed in a unit circle.

5. Now use your square to build an octagon as in Figure 2.

(a) Argue that the circumference of the circle is larger than the perimeter of the octagon.

(b) Compute the lengths of the segment a_8 joining the midpoint of the edge of the octagon to the center of the circle, and the segment b_8 forming half the edge of the octagon. Hint: I would first compute the edge $2b_8$ of the octagon using a right triangle. I would then find another right triangle and compute a_8.

(c) Compute the perimeter of the octagon. You know that this perimeter must be less than the circumference 2π of the circle. What estimate does this give you for π?

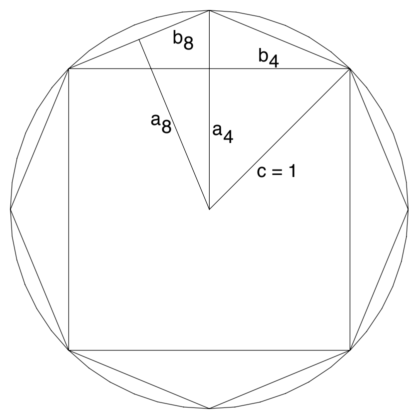

FIGURE 2. An octagon inscribed in a unit circle.

6. Try to repeat these calculations with a 16-sided polygon or a 32-sided polygon. Exactly the same arguments with right triangles should let you compute the perimeters of these polygons and get quite good estimates of π.

If you are more ambitious, turn this into a general method that lets you compute the perimeter of a 64-gon, a 128-gon, and so on. One of my kids and I once computed 10 or 12 digits of π this way. It's thrilling to me that one can do this rather easily as long as one can compute square roots accurately.

More impressively, several early Japanese mathematicians (Muramatsu Shigekyo, and Isomura Yoshinori in the 17th century, among others) used exactly this method to get large numbers of digits of π by hand.

You might also try seeing what happens if you use circumscribed polygons instead of inscribed polygons. This would let you actually bracket π between upper and lower bounds.

Better still, a number of years ago, Brian Faye, one of my students in this class, took this lab and obtained a very elegant formula for π using nested radicals, whose full meaning is still not clear to me, and which does not seem to be widely known, though it is not new. He did this by just doing the problems in this lab, but working symbolically instead of numerically, as I had always done. I was enormously impressed by his cleverness and his willingness to use computer algebra systems to their full advantage.