

CALCULUS A, LAB 3

In this lab, I'd like to explore both graphically and algebraically the process of estimating the slopes of tangent lines to curves, and to tie this in with limits and derivatives. Again, please work in pairs if you can.

Notice that this lab is written on 2-sided paper.

As always, please explain your reasoning, and please write to be read. Use *Sage* or the *Grapher* or other tools where they help, and avoid them where they hinder.

Let f be the function given by the rule

$$f(x) = \frac{60x}{x^2 + 21}.$$

We'll be doing a lot with this function, so it might make sense to define it in *Sage* by doing something like

$$f(x) = 60 * x / (x \wedge 2 + 21).$$

We'll be interested in the function f and in the slope of the tangent line to $y = f(x)$ at $x = 3$, that is, at the point $(3, 6)$.

1. Make a careful plot of the graph of f and sketch (I'd do it by hand) the tangent line to the graph at the point $x = 3$. Estimate the slope of this line from the graph. Please be principled and stick to your estimate once you have made it. If you're using *Sage* and you want to make the scale the same on the x and y axes, you can say something like

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plot(60*x/(x^2+21), (x, -2, 15)).show(aspect_ratio=1)
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2. Find the y coordinate of the point on the graph of f with $x = 4$. Sketch the line joining this point to our original point $(3, 6)$, and compute the slope of this line. Repeat this process with the line connecting $(3, 6)$ to the point on the graph with $x = 6$, the line connecting $(3, 6)$ to the point on the graph with $x = 10$, the line connecting $(3, 6)$ to the point on the graph with $x = 2$, and the line connecting $(3, 6)$ to the point on the graph with $x = 0$. Which of these lines best approximates the tangent line at the point $(3, 6)$?
3. By just getting numerical values of the slopes of secant lines, try to get an accurate (say, accurate to 3 digits) approximation to the slope of the tangent line at the point $(3, 6)$. How well does this approximation agree with your initial guess in Problem 1?
4. Now do an exact algebraic computation of the slope of the tangent line to $y = 60x/(x^2 + 21)$ at the point $(3, 6)$. To do this,
 - (a) Write an equation for the slope of the line joining the point $(3, 6)$ to the point at $(x, 60x/(x^2 + 21))$. Be as explicit as you can.
 - (b) Write down a limit whose value is the slope of the tangent line to $y = f(x)$ at the point $(3, 6)$.
 - (c) Evaluate this limit analytically.

Your exact calculation here should be in reasonable agreement with the approximate computation in Problem 3 and with your graphical estimate in Problem 1.

5. Finally, once you have the slope of the tangent line, write the equation of this line. Plot the line and the original function on the same axes. Do they look tangent?
6. Should you have some additional time, here's a problem you might want to think about for your own enlightenment, or for extra credit.

To compute the slope of the tangent line to the curve $y = f(x)$ at the point $x = a$, we have been taking the secant line joining the point $(a, f(a))$ and a second point $(x, f(x))$. The slope of this secant line is

$$\frac{f(x) - f(a)}{x - a}.$$

The secant line will swing closer and closer to the tangent line as $x \rightarrow a$, so the slope of the tangent line is

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

This tangent line slope is denoted $f'(a)$.

Sometimes people use different names for our two points. They set out to compute the slope of the tangent line at the point $(x, f(x))$, and they let the second point be a point a distance h away, the point at $(x+h, f(x+h))$. With this notation the slope of the secant line is

$$\frac{f(x+h) - f(x)}{h},$$

and the slope of the tangent line is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Which of these two formulas for the derivative one uses is a matter of choice. They're equivalent, but sometimes one or the other may result in what looks like an easier calculation. Also, the second formula might make it a bit clearer that f' is a function in its own right, which it ought to be: at each point x , the function $y = f(x)$ has a different slope.

- (a) Use this formula to compute the derivatives of x , x^2 , x^3 , and x^4 . Feel free to use *Sage* in expanding out the difference quotient $(f(x+h) - f(x))/h$, but then work out the limit yourself; don't just use *Sage's* **limit** command.
- (b) For each of the functions y above, plot y and y' on the same graph. Does it make sense that the slope of y should be the height of y' ?
- (c) Can you conjecture anything about the derivative of x^n for an arbitrary n ?
- (d) Does your conjecture produce the right answer for the derivative of $x^{-1} = 1/x$? How about $x^{-2} = 1/x^2$? $x^{1/2} = \sqrt{x}$? $x^{1/3} = \sqrt[3]{x}$? How about x^π ? In many cases, you should be able to compute the limits to get these derivatives exactly. In other cases, like x^π , you may still be able to gather numerical evidence for your conjecture.
- (e) How sure are you your conjecture works for every real number n ? For what real numbers n are you *certain* it works?