

## CALCULUS A, LAB 5 TAYLOR/MACLAURIN POLYNOMIALS

Now that we have all this machinery for computing derivatives, I thought it would be fun to look ahead for a moment at one of the big uses will find in Calc B for derivatives and second derivatives and third derivatives and so on.

So far this semester, the goal for the course could be summarized by saying that we have been trying to approximate arbitrary functions using straight lines. These best-fit straight lines are the tangent lines. The best straight line approximation to the curve  $y = f(x)$  near the point  $x = 0$  ought to be the tangent line,

$$(1) \quad y = f(0) + f'(0)x.$$

In the 18th century, when not doing things like unsuccessfully defending Edinburgh from the forces of his rightful sovereign led by Prince Charles Edward Stuart, my kinsman Colin Maclaurin investigated the question of how we might approximate arbitrary functions near  $x = 0$  using not straight lines but parabolas or cubic curves or in general polynomials of any degree. Around the same time, Brook Taylor, from a thieving country to the south of Scotland, made an obvious generalization of Maclaurin's work by approximating functions near any point, not just near  $x = 0$ . As a result of Taylor's minor improvement, these polynomials are generally known as Taylor polynomials—one more reason to lament that the Highlanders failed to press on to London in 1745, but perhaps I digress.<sup>1</sup>

How might one extend our work on tangent lines to obtain approximating polynomials of higher degree? Well, one way to describe the tangent line (1) is as the unique line that has the same value and derivative as  $f$  at the point  $x = 0$ . That is, if we let

$$(2) \quad T_1(x) = f(0) + f'(0)x,$$

then  $T_1(0) = f(0)$  and  $T_1'(0) = f'(0)$ .

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<sup>1</sup>Obviously I'm joking. Maclaurin was indeed a brilliant guy. He was admired by Newton, and until 2008 he was the youngest person ever to become a university professor. (He started as a student at the University of Glasgow at age 11, and became a professor at Aberdeen at 19.) Taylor, however, also did very important work, referenced by Maclaurin in his own writings. Further, the series studied by Maclaurin and Taylor had been floating about for some time. Gregory had used them in the 17th century, and already in 14th century Kerala in South India, Madhava knew the Taylor series for various trig functions. It is not impossible that Taylor and Maclaurin may have been guided by reports of Madhava's work that might have reached Europe.

On the political aside, while the Jacobite rebellions abound in the romance of a lost cause, none of Prince Charles' modern admirers has actually had to live under a Stuart absolute monarch, something that was no picnic. I honor the valor of my Jacobite clansmen, and I look with great pride on their struggle, which really did bring a force of Gaels within striking distance of London; but realistically, it's pretty hard to argue that installing in London or in Edinburgh in the 1740s a Stuart absolute monarch in the French style would really have been A Good Thing™. Boy, it would have been emotionally satisfying, though, in the brief moment between the return of the king and the descent of Britain and its empire into a *bellum omnium contra omnes*.

Let's do math.

One natural way to try to approximate  $f(x)$  with a parabola near  $x = 0$  would be to use a quadratic polynomial having the same value, the same derivative, and the same second derivative as  $f$  at  $x = 0$ .

1. Show that

$$(3) \quad T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

is this polynomial. That is, show that if  $T_2(x)$  is given by equation (3), then  $T_2(0) = f(0)$ ,  $T_2'(0) = f'(0)$ , and  $T_2''(0) = f''(0)$ .

The polynomials  $T_1(x)$  and  $T_2(x)$  are called the Taylor polynomials at  $x = 0$  of degrees 1 and 2, resp. Taylor polynomials at  $x = 0$ , the only kind we'll look at in this lab, are also called Maclaurin polynomials. I'll use this name both to honor my clansman and to avoid having to say "at  $x = 0$ " again and again. I'll denote them with  $T$ , though, out of fairness to Taylor.

The tangent line given by  $T_1(x)$  in equation (2) has the same height and slope as  $f$  itself at the point  $x = 0$ . What adding in the quadratic term  $\frac{1}{2}f''(0)x^2$  in  $T_2(x)$  does is to allow us to try to match the curvature of  $y = f(x)$  near  $x = 0$ .

2. Compute the first and second degree Maclaurin polynomials for the functions  $f(x) = x - \cos x$  and  $g(x) = \ln(1 + x)$ . Plot these polynomials together with the original functions and notice how the added quadratic term improves the approximation.

OK, we (and Taylor, and Maclaurin, and Gregory, and Madhava) are now on a roll. We can approximate functions not just with straight lines, but with parabolas that fit the functions even better. Can we push this even farther? What if we try to use cubic or quartic polynomials instead of quadratics?

3. Let  $f(x)$  be any function. (Don't use some particular function; just use  $f(x)$ .) Suppose we want to approximate  $f(x)$  near  $x = 0$  with a polynomial

$$T_5(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$$

of degree 5. What do  $a_0, a_1, a_2, a_3, a_4, a_5$  need to be in order for the first 5 derivatives of  $T_5$  at  $x = 0$  to agree with the first 5 derivatives of  $f$  at  $x = 0$ ? *Hint:* Just compute the derivatives and set them equal to one another and solve. You already know that  $a_0 = f(0)$ , that  $a_1 = f'(0)$ , and that  $a_2 = \frac{1}{2}f''(0)$ ; you can expect similar results for  $a_3, a_4$ , and  $a_5$ .

Armed with the calculation from Problem 3, we now seem to have a sequence of higher and higher degree polynomials approximating  $f$  near  $x = 0$  and given by

$$\begin{aligned} T_0(x) &= f(0) \\ T_1(x) &= f(0) + f'(0)x \\ T_2(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 \\ T_3(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 \end{aligned}$$

and in general

$$(4) \quad T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k.$$

These are the Maclaurin polynomials, the Taylor polynomials at  $x = 0$  (sorry, Gregory and Madhava).

4. Suppose  $f(x)$  itself is just a polynomial. What are its Maclaurin polynomials? One way to find out is just to pick some random polynomial of degree 4 or so and to work out the Maclaurin polynomials. When you do, you'll see what's happening.

For many functions, the Taylor polynomials are easier to compute than you might expect. For instance, if  $f(x) = e^x$ , then every derivative  $f^{(k)}(x) = e^x$ , so every  $f^{(k)}(0) = 1$ . The Maclaurin polynomial of degree  $n$  for  $e^x$  at  $x = 0$  is therefore

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^n \frac{x^k}{k!}.$$

5. Find patterns like this in the derivatives at 0 that let you write down the Maclaurin polynomial of degree  $n$  for the functions
- (a)  $f(x) = \sin x$ .
  - (b)  $f(x) = \cos x$ .
  - (c)  $f(x) = \frac{1}{x+1} = (x+1)^{-1}$ .
  - (d)  $f(x) = \ln(x+1)$ .

I don't care whether you write the answers in  $\Sigma$ -notation, or whether you just write enough terms that the pattern becomes clear.

6. Computing the Maclaurin polynomials for

$$f(x) = \frac{1}{x^2 + 1}$$

is a pain, because when you start taking derivatives, the quotient rule gives you messier and messier results. (Try it and see.) Despite this, you can get good polynomial approximations for  $1/(x^2+1)$  (which in fact turn out to be the Maclaurin polynomials, though we won't prove this) by just taking the Maclaurin polynomials for  $1/(x+1)$  that you computed in Problem 5, part (c) and replacing  $x$  by  $x^2$ . Do this. *Remark:* This problem really is as stupidly easy as it sounds.

Our goal has been to approximate the function  $f(x)$  really well close to the point  $x = 0$ . The Madhava/Gregory/Taylor/Maclaurin idea is to get the shape of our polynomial curves very close to the shape of the function  $f(x)$  itself near  $x = 0$  by getting more and more derivatives correct right at the point  $x = 0$ . A question to ask, though, is how well these polynomials approximate the function  $f(x)$  when we move away from  $x = 0$ .

7. Plot the function  $f(x) = \cos x$  and a few of its Maclaurin polynomials, like  $T_2(x)$ ,  $T_4(x)$ ,  $T_6(x)$ ,  $T_8(x)$ ,  $T_{10}(x)$ , and perhaps some larger one. For what values of  $x$  does it look like that Maclaurin polynomials eventually get close to the graph of  $f(x)$ ? I find this plot really stunning. Do you?
8. Do the same thing with the function  $f(x) = 1/(x+1)$  from Problem 5, part (c) and with the function  $f(x) = \ln(x+1)$  in Problem 5, part (d). For what values of  $x$  does it look like the Maclaurin polynomials are converging to the function itself?

9. Do the same thing with the approximating polynomials we found for the function  $f(x) = 1/(x^2 + 1)$  from Problem 6. Where does it look like these polynomials are converging to the function itself?

This is really a stretch, asking you to generalize based on way too little data, but is there anything you can conjecture about where in general Maclaurin polynomials do and don't converge to the function  $f$  they approximate?

Let me add just one final comment. We've been talking about approximating functions with polynomials, but if the polynomials just get closer and closer and closer to the value of the function, then it would be reasonable not just to add up finitely many terms in a Taylor/Maclaurin polynomial, but to add up all the infinitely many terms we encounter in all these polynomials. That is, instead of looking at the polynomial

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k,$$

why not look at the infinite series (called a Taylor or Maclaurin series)

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k?$$

Where the Taylor polynomials converge to the value of  $f$ , we might expect the infinite Taylor series to give not just an approximation to  $f(x)$ , but an exact value for  $f(x)$ . We'll see in Calc B that this is right. For instance, we'll see that for every  $x$ ,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Our Maclaurin series for sine and cosine also give exact values for these functions. This is, in fact, what your calculator does to give you values of the trig functions: it adds up enough terms of the Maclaurin series to be certain that all the digits displayed are correct. Amazing! We don't have to approximate  $\sin \theta$  by carefully drawing a big circle and measuring the  $y$ -coordinate of the point at angle  $\theta$ . We now have an actual formula for  $\sin \theta$ .

When we set out to compute derivatives, our program really had three steps. First, we gave up, admitting that we didn't know how to compute slopes of curves. Second, having given up getting an exact value for the slope, we tried to get approximate slopes using secant lines. Third, we turned surrender into victory by taking limits of our approximate slopes to get an exact slope.

We've done the same thing here, haven't we? We didn't know how to compute the trig functions. We therefore gave up on that and tried to approximate them with polynomials, only to find that a limit of the polynomials gives us the exact values we had despaired of finding. Our Three Step Program has worked again!

This won't be the last time we will work the Three Steps, nor the last time that program will work for us. Keep coming back!