CALCULUS A
LAB 6

Introduction. We've been looking at one approach to computing the area

$$\int_a^b f(x)\,dx$$

under a curve: we divide $[a,b]$ into some number n of narrow slices. We then approximate the area within each slice and add up these approximations for all the slices. Finally we take a limit as $n \to +\infty$ in order to get the exact area.

Now, there is an obvious problem with this approach, which we've already seen in class. Normally we can't simplify the Riemann sums we compute into some closed form, which means that we certainly can't take the next step and take a limit.

Those of you who have seen calculus before and remember the machinery might think that there is an easy way around this problem: We compute the area by taking the antiderivative of f and then using the Fundamental Theorem of Calculus. Unfortunately, it is also often impossible to compute antiderivatives; so this method, too, often fails to deliver up an exact answer for the area.

The result is that when computing areas in real life, one is often reduced to settling for an approximation gotten by adding up the areas of a finite number of slices.

The purpose of this lab is to look for ways to minimize our numerical work and to maximize the accuracy of our approximations if we are reduced to adding up slices.

Methods for approximating areas. Suppose you want to compute the area under the graph of $y = f(x)$ between $x = 0$ and $x = 3$. Suppose, further, that you can't work out the result analytically, and that you need to compute approximate numerical areas of a bunch of slices and to add up the results. For concreteness, suppose you decide to cut the interval $[0,3]$ into 4 slices. Then I can think of at least 4 methods you might use to approximate the areas.

(1) **Right-hand rectangles.**

In this method, which we've used in class, we approximate f in each slice by a constant function whose height is the value of f at the right hand end of each slice. The area we're computing is then shown in Figure 1. Algebraically, the Riemann sum is

$$\sum_{k=1}^{4} f\left(\frac{3k}{4}\right)\cdot\frac{3}{4} = \left\{ f\left(\frac{3\cdot 1}{4}\right) + f\left(\frac{3\cdot 2}{4}\right) + f\left(\frac{3\cdot 3}{4}\right) + f\left(\frac{3\cdot 4}{4}\right)\right\}\cdot\frac{3}{4}.$$

(2) **Left-hand rectangles.**

Here we use the same idea, except we approximate f in each slice by a constant function whose height was the value of f at the left hand end

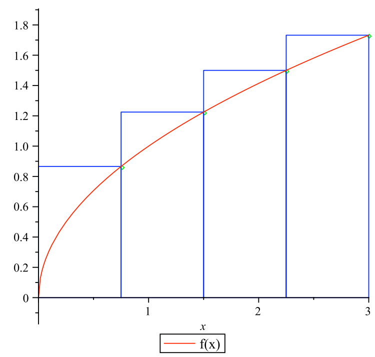

FIGURE 1. Right-hand Riemann sum.

of each slice. The area we're computing is then shown in Figure 2. Algebraically, the Riemann sum is

$$\sum_{k=0}^{3} f\left(\frac{3k}{4}\right)\cdot\frac{3}{4} = \left\{f\left(\frac{3\cdot 0}{4}\right)+f\left(\frac{3\cdot 1}{4}\right)+f\left(\frac{3\cdot 2}{4}\right)+f\left(\frac{3\cdot 3}{4}\right)\right\}\cdot\frac{3}{4}.$$

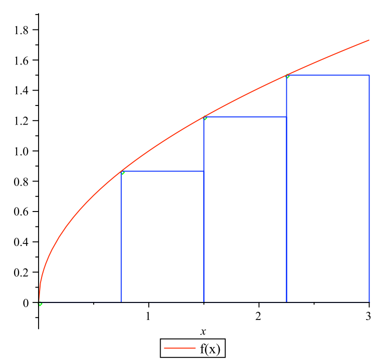

FIGURE 2. Left-hand Riemann sum.

(3) **Midpoints.**
A third method we have at least written down is to use the value of the function at the midpoint of each slice as the height of the rectangle. That is, instead of using either the right-hand or left-hand endpoint of each slice, we use the midpoint. The area we're computing is then shown in Figure 3. Algebraically, the Riemann sum is

$$\sum_{k=0}^{3} f\left(\frac{3k}{4}+\frac{3}{8}\right)\cdot\frac{3}{4} = \left\{f\left(\frac{3\cdot 0}{4}+\frac{3}{8}\right)+f\left(\frac{3\cdot 1}{4}+\frac{3}{8}\right)+f\left(\frac{3\cdot 2}{4}+\frac{3}{8}\right)+f\left(\frac{3\cdot 3}{4}+\frac{3}{8}\right)\right\}\cdot\frac{3}{4}.$$

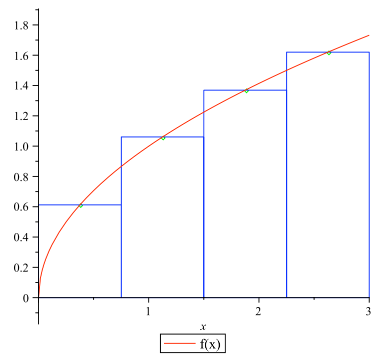

FIGURE 3. Midpoint Riemann sum.

(4) **Trapezoids.**
A final method that seems very appealing is to approximate the areas by putting trapezoids in each slice. We could write down a new formula for this area, but there is a simpler way to think about it. In each slice, the area of the trapezoid is midway between the area of the left-hand rectangle and the area of the right-hand rectangle. The same thing has to work if we add up all the trapezoids: the area has to be the average of the areas of the right-hand and left-hand rectangles. (If you don't believe me, draw the picture showing both the left-hand and right-hand sums together with the trapezoids.)

General formulas. In general, if we want to approximate

$$\int_a^b f(x)\,dx$$

using n slices, then the right-hand rectangle method would tell us to compute the sum

$$\sum_{k=1}^{n} f\left(a + \frac{(b-a)k}{n}\right)\frac{b-a}{n}.$$

The left-hand rectangle method says to compute

$$\sum_{k=0}^{n-1} f\left(a + \frac{(b-a)k}{n}\right)\frac{b-a}{n}.$$

The trapezoid method is the average of these two sums. The midpoint method says to compute

$$\sum_{k=0}^{n-1} f\left(a + \frac{(b-a)(k+\frac{1}{2})}{n}\right)\frac{b-a}{n}.$$

All these formulas are written in *Sage* form on the attached *Sage* worksheet.

Example. Suppose we want to compute the area under the sine function between $x = 0$ and $x = \frac{\pi}{2}$,

$$\int_0^{\pi/2} \sin x \, dx.$$

We've never done a problem like this in class, so it looks like we'll have to approximate.

(Actually, of course, the situation isn't hopeless. If we believe what we have conjectured about velocity and distance, that the area

$$\int_a^b f(x) \, dx$$

could be gotten by letting F be an antiderivative of f and then computing $F(b) - F(a)$, then

$$\int_0^{\pi/2} \sin x \, dx = (-\cos(\pi/2)) - (-\cos 0) = 1.$$

This is useful to know as we start looking at the error in our approximations. But let's go back to assuming we need to approximate.)

1. Working with the integral $\int_0^{\pi/2} \sin x \, dx$ from the example, begin by filling in the following tables, showing the approximations produced by each method and showing their errors from the exact value of 1. Follow the advice on the *Sage* worksheet to avoid throwing away precision unnecessarily.

Approximation to the area

n	Right rectangles	Left rectangles	Trapezoids	Midpoints
1				
10				
100				
1000				
$10,000$				
$100,000$				

Error in the approximation

n	Right rectangles	Left rectangles	Trapezoids	Midpoints
1				
10				
100				
1000				
$10,000$				
$100,000$				

2. Some columns contain approximations that are too large (bigger than 1), and others contain approximations that are too small (less than 1). Explain geometrically how you could have determined in advance which columns would be which.

3. **(a)** Look at the errors in the right rectangle method. They should form a rough pattern. Are they proportional to $1/n$ or to $1/n^2$, or what? If you needed to work out the integral to 100 digit accuracy using the right rectangle method, roughly how big would n have to be? An order of magnitude estimate is fine. Don't try actually getting that level of accuracy in the computation; you might not like the wait.

(b) Repeat part (a) for the left rectangle method.

(c) Repeat part (a) for the trapezoid method.

(d) Repeat part (a) for the midpoint method.

4. Compare the errors of the midpoint and trapezoid methods. Which worked better? Is there any relation between the sizes and signs of the errors in the two methods?

5. Exploit the connection in Problem 4 to suggest a way to combine the midpoint and the trapezoid method into a new method more accurate than either alone. Try out the new method on our sample integral, and repeat problem 3(a) with the new method. Don't quit until you have a method which is way better than the midpoint method. Just twice as good is not enough.

6. Once you have a really good method in Problem 5, try it out on a quadratic function. See if you can prove anything.