Name Tim McLarnan

CALCULUS A, TEST 3, 2011

This test is to be done working alone, and to be returned to the Library within 3 hours of the time you initially check it out from the Library.

You may use a calculator if you wish. You're not welcome to computers. In particular, don't just use *Sage* and believe the results it gives you. Show me how to get your results.

You need not simplify the results of your calculations. As always in math, you should show me what you are doing, explaining so a reader can understand and follow your work. When I ask for numerical answers, please give me exact expressions like e^{π}, not numerical approximations like 23.140692632779269007.

Please remember that you are writing to be read.

1. (a) State the definition of derivatives in terms of limits.
   (b) Use this definition, and nothing more sophisticated, to compute $f'(2)$, where

$$f(x) = \frac{1}{2x+1}.$$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

Here, $f'(2) = \lim_{x\to 2} \dfrac{\frac{1}{2x+1} - \frac{1}{5}}{x-2} = \lim_{x\to 2} \frac{1}{x-2}\left[\frac{1}{2x+1} - \frac{1}{5}\right]$

$$= \lim_{x\to 2} \frac{1}{x-2}\,\frac{4-2x}{5(2x+1)} = \lim_{x\to 2} \frac{-2}{5(2x+1)} = -\frac{2}{25}.$$

Just as a check, we know the derivative is $\frac{-1}{(2x+1)^2}\cdot 2$, which at $x=2$ is $-\frac{2}{25}$; so we haven't messed up.

2. Show how you would compute the following quantities by hand. For derivatives, tell me what rules (Product, Quotient, Chain, etc.) you use. For integrals instance, if you use Substitution, show what substitution. For at least one of these, you can think first as a way to avoid calculating.

(a) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ $= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$

(b) The derivative of $x^4 - \frac{1}{\sqrt{x}} + \frac{1}{\ln x}$

Power rule, Chain Rule.

$4x^3 + \frac{1}{2}x^{-3/2} - \frac{1}{(\ln x)^2} \cdot \frac{1}{x}$

One could have used Quotient Rule instead of Chain Rule

(c) $\frac{d}{dx}\left(\sqrt{\sin(x^3-5x)}\right)$

Power Rule, Chain Rule twice

$\frac{1}{2\sqrt{\sin(x^3-5x)}} \cdot \cos(x^3-5x) \cdot (3x^2-5)$

(d) $f'(x)$, when $f(x) = \cos x \tan x$

I'd say $\cos x \tan x = \sin x$, whose derivative is $\cos x$.

There are other approaches.

(e) $\frac{d}{dx}\left(\frac{x^2+1}{x\ln x}\right)$ $\qquad \dfrac{(x\ln x)2x-(x^2+1)(\ln x+\frac{x}{x})}{x^2(\ln x)^2}$

Quotient Rule, then Product Rule.

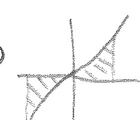
(f) $\int_1^4\left(x^2+\frac{1}{\sqrt{x}}\right)dx = \left[\frac{x^3}{3}+2\sqrt{x}\right]_1^4 = \frac{64}{3}+4-\frac{1}{3}-2 = 23$

(g) $\int \frac{\cos(\ln x)}{x}\,dx$

Let $u=\ln x$, $du=\frac{1}{x}dx$. This integral then becomes $\int \cos u\,du = \sin u + c$
$= \sin(\ln x)+c$

(h) $\frac{d}{dx}\int_2^x t^2e^t\,dt = x^2e^x$, by the useless form of the FTC.

(i) $\int_{-1}^1 \tan x\,dx.$ Tan x is an odd function, so the integral is 0



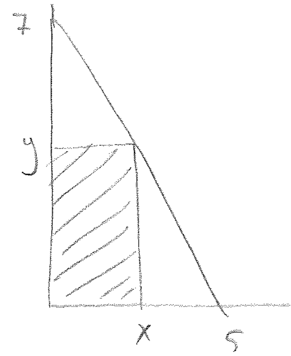
Alternatively, let $u=\cos x$, $du=-\sin x\,dx$. The indefinite $\int$ is then
$-\int\frac{1}{u}\,du = \ln|u|+c = \ln|\cos x|+c$. The definite $\int$ is then $\ln(\cos 1)-\ln(\cos 1)=0$.

3. What do the two parts of the Fundamental Theorem of Calculus say? Try to be as algebraically precise as you can.

First, if f is continuous on $[a,b]$ and $x \in (a,b)$, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Second, if $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

4. I own a parcel of land forming a triangle with vertices at $(0,0)$, $(5,0)$, and $(0,7)$ (It doesn't matter but distances are in furlongs, a quaint Imperial unit. 1 furlong is 220 yards or 1/8 mile.) I want to put a rectangular corral on my land, and because of my beliefs about feng shui, I would like its sides to be parallel to the coordinate axes. What are the dimensions of the corral of maximum area?



I want to maximize xy on $0 \leq x \leq 5$ subject to the constraint that $y = 7 - \frac{7}{5}x$.

That is, I'm maximizing $7x - \frac{7}{5}x^2 = A(x)$ on $[0, 5]$.

$A'(x) = 7 - \frac{14}{5}x = 0 \iff x = \frac{7 \cdot 5}{14} = \frac{5}{2}$.

This point is obviously the global max, since there are no points where $A'(x)$ is undefined, and since $A(0) = A(5) = 0$.

The maximum area is $\therefore$ when $x = \frac{5}{2}$, $y = \frac{7}{2}$, $A = \frac{35}{4}$.

Of course, 1 square furlong = 10 acres, so the maximum area is 87.5 acres.

2. Let the function f and its derivatives be given by

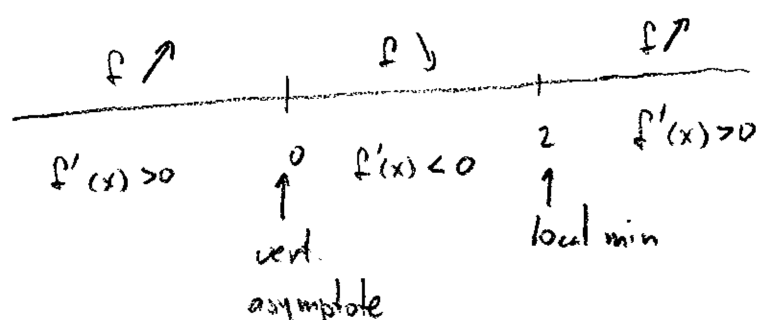
$$f(x) = x + \frac{1}{x} + \frac{3}{x^2} = \frac{x^3+x+3}{x^2}$$

$$f'(x) = 1 - \frac{1}{x^2} - \frac{6}{x^3} = \frac{(x-2)(x^2+2x+3)}{x^3}$$

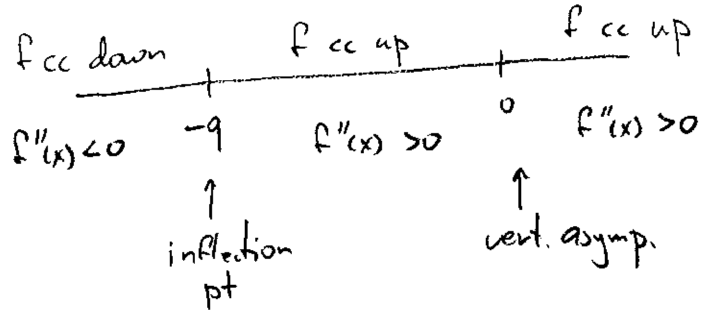
$$f''(x) = \frac{2}{x^3} + \frac{18}{x^4} = \frac{2x+18}{x^4}.$$

(a) Where is f increasing? Decreasing? What x values are local maxima? Minima?
(b) Where is f concave up? Down? What x values are inflection points?
(c) To what very simple function is $f(x)$ asymptotic when $|x|$ gets large?
(d) Give a rough sketch of $y = f(x)$, showing important features you identified in the previous parts of the problem and showing anything else you think is important.

(a) $f'(x) = 0$ when $x = 2$, and $f'(x)$ is undefined at the vertical asymptote at $x=0$. (The quadratic x^2+2x+3 has no real roots.) What f' tells us is then:



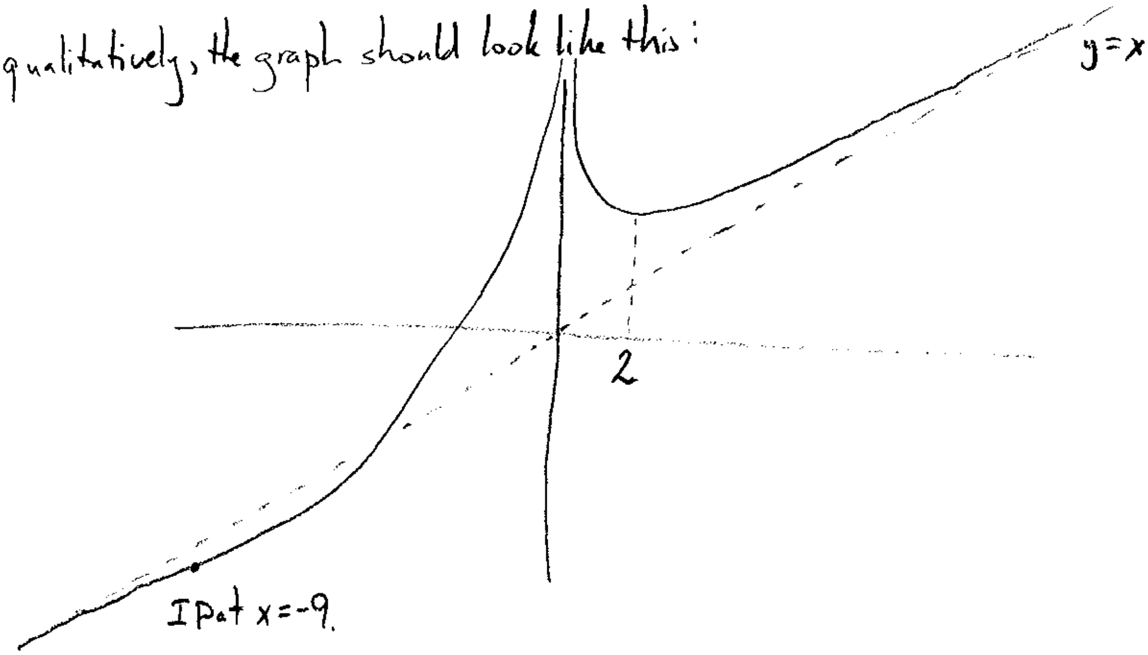
(b) $f''(x) = 0$ when $x = -9$, and $f''(x)$ is undefined at the vertical asymptote at $x=0$.
The 2nd derivative then fore tells us:



(c) When $|x|$ is large, both $\frac{1}{x}$ and $\frac{1}{x^3}$ are tiny, and $f(x) \approx x$.

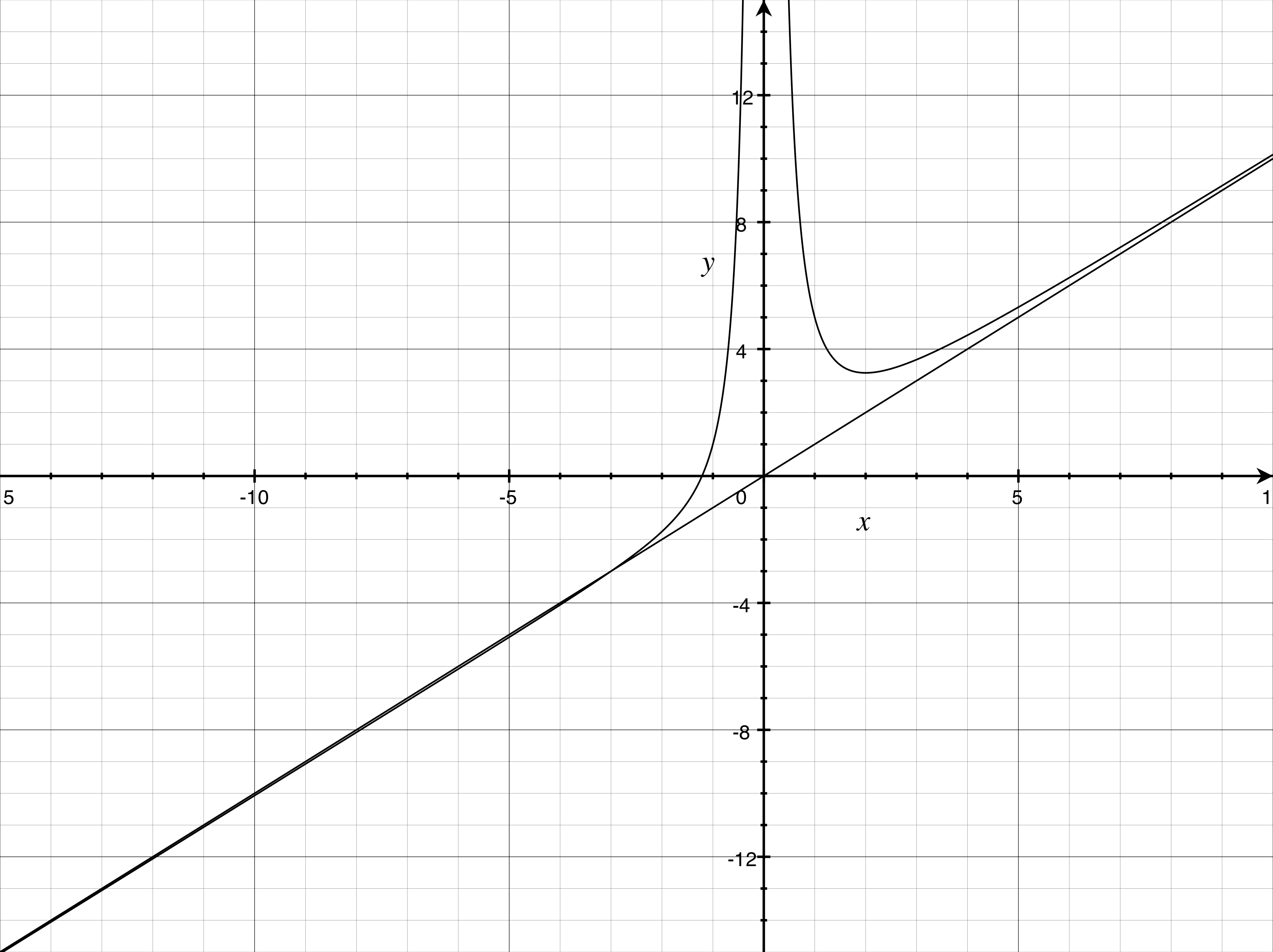
More Problem 2

(d) So qualitatively, the graph should look like this:



In fact, the inflection point at $x=-9$ is impossible to spot in the plot.

I've attached a plot from the Apple Grapher.



y
x
12
8
4
0
-4
-8
-12
5
-10
-5
5
1