

**DISCRETE MATHEMATICS**  
**HOMEWORK 1**

Here are a few problems to think about between now and Friday, and to write up between now and next Wednesday. Try to think creatively about the questions, and to convince the class of your answers.

1. Let the system  $\mathbb{Z}_5$  denote the “clock arithmetic” consisting of the numbers 0, 1, 2, 3, 4. Arithmetic is done just like they taught you in grade school, except that whenever the answer is 5 or greater, you subtract off copies of 5 until it is small enough. For instance,  $2 + 2 = 4$ , but  $3 \cdot 4 = 12 = 12 - 5 = 7 = 7 - 5 = 2$ .

Find the following elements of  $\mathbb{Z}_5$ , if they exist:  $-1$ ,  $-2$ ,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{5}$ ,  $\sqrt{2}$ ,  $\sqrt{-1}$ . Do the same thing in  $\mathbb{Z}_7$  and in  $\mathbb{Z}_6$ . Can you say anything about these numbers in  $\mathbb{Z}_n$  for an arbitrary integer  $n$ ?

2. Suppose you know about the integers (the whole numbers), but you don't know much else about mathematics. You want to talk about when one integer divides another. You decide to use the notation  $a \mid b$  to mean “ $a$  divides  $b$ ” (i.e., “ $b$  is a multiple of  $a$ ”). The symbol  $a \mid b$  is not a numerical value, but a true-false assertion. For instance,  $3 \mid 6$  is a true statement, and  $3 \mid 5$  is a false statement.
  - (a) Give a definition of  $a \mid b$  in mathematical terms.
  - (b) Convince me of the truth or falsehood of the assertion that for all integers  $a$ ,  $b$ ,  $c$ , if  $a \mid b$ , then  $a \mid bc$ .
  - (c) Convince me of the truth or falsehood of the assertion that for all integers  $a$ ,  $b$ ,  $c$ , if  $a \mid b$  and  $a \mid c$ , then  $a \mid (b + c)$ .
  - (d) Convince me of the truth or falsehood of the assertion that for all integers  $a$ ,  $b$ ,  $c$ , if  $a \mid bc$ , then  $a \mid b$  or  $a \mid c$ .

*Remark:* You may not have met the notation  $a \mid b$  before, but it is standard notation among mathematicians. It is vaguely related to  $a/b$ , but it is not the same thing. For instance,  $6/3$  evaluates to 2, and  $3/6$  evaluates to 0.5; but  $6 \mid 3$  evaluates to *false*, and  $3 \mid 6$  evaluates to *true*.

It shouldn't be particularly surprising that the direction of the bar matters. After all, it probably doesn't bother you that  $3 = 6 - 3 \neq 6/3 = 2$  or that  $-3 = 3 - 6 \neq 3/6 = 0.5$ . Why should it bother you that *true*  $= 3 \mid 6 \neq 3/6 = 1/2$ ?

3. You probably learned in school that every positive integer can be factored into primes (numbers whose only positive divisors are themselves and 1), and that there is only one such factorization for every number. That is, every positive integer  $n$  can be written in exactly one way as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where  $p_1, p_2, \dots, p_k$  are primes.

- (a) The function  $\tau(n)$  counts the positive divisors of  $n$ . For instance,  $\tau(7) = 2$ , since 7 has 2 positive divisors (1 and 7). Similarly,  $\tau(10) = 4$ , since 10 has 4 positive divisors (1, 2, 5, and 10), and  $\tau(9) = 3$ , since 9 has the 3 positive divisors 1, 3, and 9.

Try to find a formula for  $\tau(n)$  in terms of the prime factorization of  $n$ . That is,

$$\tau(n) = \tau(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = ?$$

The formula you find is likely to involve the numbers  $p_i$  and/or  $\alpha_i$  in some manner or other.

- (b) The problem says that every positive integer factors into primes in exactly one way. How sure are you that this is true? Could you give a persuasive argument why every positive integer factors into primes? That it could not have 2 different factorizations into primes?