DISCRETE MATHEMATICS
HOMEWORK 10

This homework is one of the larger assignments or take-home tests which collectively count for 60% of your grade. Please take it seriously; please don't take it too seriously. You are welcome to use your notes, including any handouts I've given you. You are welcome to use computing devices for computing. You may not use other references or discuss the test with other people (except for prayer to the saints, which the test is designed to encourage). If you are uncertain about whether something is allowed, why not ask me?

I've tried to make the test clear and accurate, but nothing in this life comes with a guarantee. Feel free to call me at work or at home (but not after 9 PM, please), or to send e-mail to `timm` if you have questions. If I find errors on the test, I will send e-mail to the class; so looking at your mail now and then might be wise.

This test is due in class on Monday. I am tolerant of late homework, but not of late tests.

As is almost always the case on my tests, some of the problems are routine, and some are more challenging. Don't panic if there are problems you can't do; nobody in life bats anywhere near 1.000. Do the ones you can do, make a good effort to stretch for the hard ones, stay calm when some of them get past you. But do try a bit of numerical experimentation before you pack any in; focused play is a tool of great power. Remember also that in mathematics, as in any other discipline, no answer is complete without a justification. A mathematical question can never be answered with "42," or "No," unless there is a calculation or argument to show the correctness of these claims.

1. Find all the integer solutions to the Diophantine equation $2013x - 902y = 66$. Convince me that all the solutions you find are correct, and that you have found all the solutions. (I know we've done this sort of problem before, but humor me and fill in the details one last time, OK?)

2. Find all the simultaneous solutions in the integers to the following pairs of congruences, and convince me you have found all solutions. Feel free to make reference to Problem 1 instead of proving the same things again.

(a) The congruences

$$x \equiv 28 \pmod{45}$$
$$x \equiv 113 \pmod{400}.$$

(b) The congruences

$$x \equiv 20 \pmod{45}$$
$$x \equiv 112 \pmod{400}.$$

3. Prove inductively that for every positive integer n,

$$(1 \cdot 2 \cdot 3) + (2 \cdot 3 \cdot 4) + (3 \cdot 4 \cdot 5) + \cdots + (n(n+1)(n+2)) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

I really want an inductive proof, even if you know another way to prove the formula.

If you get an algebraic mess, then think a bit. You should be able to avoid any algebraic nastiness.

4. Is $\sqrt{1/5}$ rational or irrational? Prove your claim.

5. Prove or disprove and salvage if possible: In every modular arithmetic $\mathbb{Z}_m$, if $ab = 0$, then $a = 0$ or $b = 0$.

To be clear: The instructions here mean that you must first either prove the result or present a counterexample. If the result is false, then try to find a result that is similar but true and that is interesting, and to prove this result.

6. Use Fermat's test to show whether 133 is a prime. To be clear: I know the answer, but I'd like to see it done by Fermat's test.

Could Fermat's test be used to show whether 59 is a prime? You need not do the arithmetic; just tell me what you would do, or explain why nothing you could do would work in this case.

7. An arithmetic system has elements a, b, c, d, e, f, g, h with addition and multiplication defined by the following rules:

$+$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	c	d	a	g	h	f	e
c	c	d	a	b	f	e	h	g
d	d	a	b	c	h	g	e	f
e	e	g	f	h	a	c	b	d
f	f	h	e	g	c	a	d	b
g	g	f	h	e	b	d	c	a
h	h	e	g	f	d	b	a	c

$\cdot$	a	b	c	d	e	f	g	h
a	a	a	a	a	a	a	a	a
b	a	b	c	d	e	f	g	h
c	a	c	a	c	a	a	c	c
d	a	d	c	b	e	f	h	g
e	a	e	a	e	c	c	f	f
f	a	f	a	f	c	c	e	e
g	a	g	c	h	f	e	d	b
h	a	h	c	g	f	e	b	d

(a) Is there a number in this system you would be tempted to call 0? Why or why not?
(b) Is there a number in this system you would be tempted to call 1? Why or why not?
(c) Which elements x in this system have additive inverses $-x$? What are these inverses?
(d) Which elements x in this system have multiplicative inverses $1/x$? What are these inverses?
(e) Which elements x in this system have square roots $\sqrt{x}$? What are these square roots?