

DISCRETE MATH
HOMEWORK 12'

At least one of you is frustrated because all the counting problems we have done so far have already been familiar. For anyone who is in this situation, here is an alternative set of problems that may be more challenging. I expect people in this situation to attempt a reasonable number of these problems in place of the simpler problems on Homework 12. Should all these still look trivial, please come talk to me so we can correct that situation.

1. The Fibonacci numbers are defined by $F_0 = F_1 = 1$, and by $F_{n+1} = F_n + F_{n-1}$ if $n \geq 1$. Prove that

$$F_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right\}.$$

One way to do this would be inductively, but it would obviously be preferable, if possible, to find a proof that better illuminates what's going on.

2. Let

$$F(x) = \sum_{n=0}^{\infty} F_n x^n$$

be the generating function for the Fibonacci numbers.

- (a) By looking at $F(x)$, $xF(x)$, and $x^2F(x)$, write $F(x)$ in a simple closed form (without the infinite sum).
- (b) Show that the sum of the numbers 0.01, 0.001, 0.0002, 0.00003, 0.000005, 0.0000008, 0.00000013, 0.000000021, ... is $1/89$.
- (c) Is it a fluke that 89 is also a Fibonacci number?
3. A *derangement* of the numbers $\{1, 2, 3, \dots, n\}$ is a rearrangement of the numbers in which no number is in its original position. That is, it's a permutation $\sigma \in \mathcal{S}_n$ in which for all i , $\sigma(i) \neq i$.

Find a formula for the number of derangements of n objects. As $n \rightarrow \infty$, what fraction of all the permutations of n objects are derangements? Inclusion-Exclusion is your friend.

(One way this problem is sometimes expressed is to say, suppose I return the homework by handing papers to people completely at random. What is the probability that nobody gets their own paper back?)

4. Show as elegantly as possible that

$$\sum_{m=k}^n \binom{m}{k} = \binom{n+1}{k+1}.$$

5. Show as elegantly as possible that

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

It would be fun to do this both algebraically and combinatorially.

6. Consider the following counting problems:

- (a) In how many ways can a set of non-intersecting diagonals of a regular n -gon be drawn which completely dissect the n -gon into triangles?
- (b) How many ways can we completely parenthesize a product of $n - 1$ factors?
- (c) How many paths are there in the plane that start at $(0, 0)$, end at $(2(n-2), 0)$, do not go below the x -axis, and in which every step in the path either goes NE (adding 1 to x and 1 to y) or SE (adding 1 to x and subtracting 1 from y)?

Show that all these problems have the same answer, and find a simple formula for that answer.