DISCRETE MATHEMATICS
HOMEWORK 14

This homework is one of the three larger assignments or take-home tests which collectively count for 60% of your grade. Please take it seriously; please don't take it too seriously. You are welcome to use your notes, including anything on the course web page. You are welcome to use computing devices. Please don't use other books, and please don't discuss the test with other living people. Prayer to the saints is, of course, fine. The test was designed to encourage this practice. Please take seriously the comments I wrote on academic integrity at the beginning of the semester. Once you start reading the problems on this test, you should not talk about mathematics with anybody until you turn the test in.

I've tried to make the test clear and accurate, but nothing in this life comes with a guarantee. Feel free to call me at work or at home (but not after 9 PM, please), or to send e-mail to `timm` if you have questions. If I find errors on the test, I will send e-mail to the class; so looking at your mail now and then might be wise.

This test is due at my office at the time of the scheduled final for the course, Monday, December 12, at or before 4:30 PM. I am tolerant of late homework, but I am almost completely intolerant of late tests. If I'm not in my office, you may slide the test under my door, give it to Bobbi in the Department Office, or put it in my mailbox in the Department Office.

As always, the different parts of the test differ substantially and capriciously in difficulty. Stay calm. Don't panic if there are problems you can't do; nobody in life bats anywhere near 1.000. Do the ones you can do, make a good effort to stretch for the hard ones, stay calm when some of them get past you. But do try a bit of numerical experimentation before you pack any in; focused play is a tool of great power.

Please continue to remember that in mathematics, no answer is complete without an explanation. I am much less interested in learning that the number of possible threesomes of US representatives in Congress is 13,624,345, than in learning where this number came from, and why it is correct.

I am perfectly content to see the numerical parts of answers written in forms like

$$\binom{435}{3} = \frac{435 \cdot 434 \cdot 433}{3!}$$

rather than in forms like 13,624,345. You don't need to multiply out answers unless you want to. The two exceptions to this principle are the Stirling numbers $\left\{ {n \atop k} \right\}$, and the partition numbers $p_k(n)$, which I would like you to evaluate either to numbers.

I assume that the formulas making up the 12-fold way are known to all of us. They need not be further justified.

Remember that you are writing to be read.

Good luck; do well; learn something new; have as much fun as possible; then have blessed holidays and come back eager to do more cool stuff.

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Do any 6 of these problems, or do all 7 for extra credit and guaranteed added delight.

1. Here are a few on binomial coefficients.

(a) The game of follyball is played by a team of 5 people: a forward, a backward, a guard, a lard, and a half-full. Play is highly specialized, so each player plays only one position. How many ways can 5 people be chosen out of our class of 21 and assigned positions to form a follyball team?

(b) The game of polyball is played by a team of 5 people, who just mill about together on the field, nobody doing anything special. How many ways can 5 people be chosen out of our class of 21 to form a polyball team?

(c) Finally, the game of Baliball is a less specialized version of follyball. It is also played by a team of 5 people, two of whom (the forward and backward) play specialized positions, and the other three of whom are all lards. How many ways can 5 people be chosen out of our class of 21 and assigned positions on a Baliball team?

2. In how many ways can 12 balls be distributed to 4 distinguishable bags if

(a) The balls are distinguishable?
(b) The balls are indistinguishable?
(c) The balls are distinguishable and each bag gets at least 1 ball?
(d) The balls are indistinguishable and each bag gets at least 1 ball?
(e) The balls are distinguishable and each bag gets 3 balls?
(f) The balls are indistinguishable and each bag gets 3 balls?

3. Repeat the previous exercise in the case the 4 bags are indistinguishable.

5. Here are a few random counting problems.

(a) In order to look like you are educated and you care about important things, you go to a place that sells books by the foot (these places actually exist; specify the color of books you want, and they send them to you). When your order is delivered, you find that ordering serious books by color means that you now own 8 identical Bibles (black), 12 identical Qur'ans (green) and 6 identical copies of Marx's *Das Kapital* (red). You've also got 2 bookshelves, each long enough to hold all your books. In how many ways can you arrange your 26 new books on your 2 shelves? Assume that on each shelf, the books are arranged starting at the left of the shelf, held in place by a single bookend at the right-hand end of the books on that shelf.

(b) A store sells 8 kinds of candy. How many ways can you pick out 15 candies total to take home in a bag?

(c) It's Christmas time at the McLarnans! The gifts this year are three identical catnip mice, 4 identical balls of yarn, one dog chew, and 4 leftover identical Bibles. In how many ways can these 12 gifts be distributed among the usual 12 distinguishable dependents (4 kids, 7 cats, and a dog) if everybody gets a gift?

(d) What if, in part (c), we replace the Bibles with Qur'ans? The difference is that in order to avoid causing offense, we feel that we cannot give Jessie the dog one of the Qur'ans.

4. We've used several times the Binomial Theorem, which says that

$$(a+b)^p = \sum_{k=0}^{p} \binom{p}{k} a^{p-k}\, b^k.$$

Another "theorem," insultingly called the Student's Binomial Theorem, says that

$$(a+b)^p = a^p + b^p.$$

Prove that if p is a prime and a and b are any numbers in $\mathbb{Z}_p$, then the Student's Binomial Theorem actually works in $\mathbb{Z}_p$.

6. I won't ask you to do a combinatorial proof, but I will ask you to show that you have the idea by taking some expressions and describing what they count.

(a) Describe something that would be counted by the expression

$$\binom{n}{k}\binom{k}{r} r.$$

(b) Describe something that would be counted by the expression

$$\frac{(a+b+c)!}{a!\, b!\, c!} \cdot a \cdot b \cdot c.$$

(c) Describe something that would be counted by the expression

$$\left\{ n \atop k \right\} k.$$

7. One way to define the binomial coefficients would have been to say four things:

$$\binom{n}{k} = 0 \text{ if } k < 0 \tag{1}$$

$$\binom{0}{k} = 0 \text{ if } k > 0 \tag{2}$$

$$\binom{0}{0} = 1 \tag{3}$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{4}$$

That is, we could require that the denominators be non-negative, that the first row of Pascal's triangle contain a single 1, and that everything else be generated by Pascal's recurrence (4).

If we used this definition, what could we say about the numbers $\binom{n}{k}$ with $n < 0$? That is, explore what $\binom{-1}{k}$ would have to be in order for rules (1)-(4) to work universally. Then look at $\binom{-2}{k}$, then $\binom{-3}{k}$ and so on. Try to see what formulas you can give. Explore playfully. Enjoy. Rejoice in the spirit of discovery you have met in this class!