DISCRETE MATHEMATICS
HOMEWORK 8

1. Find all solutions in $\mathbb{Z}$ for each of the following sets of congruences. This means you must convince me both that all the answers you find are solutions, and that no other solutions exist.

(a) The system

$$x \equiv 16 \pmod{24}$$
$$x \equiv 10 \pmod{15}.$$

(b) The system

$$x \equiv 17 \pmod{24}$$
$$x \equiv 10 \pmod{15}.$$

(c) The system

$$x \equiv 1 \pmod{4}$$
$$x \equiv 2 \pmod{5}$$
$$x \equiv 3 \pmod{17}.$$

2. In $\mathbb{Z}_{561}$, compute 2^2, 2^4, 2^8, 2^{16}, 2^{32}, 2^{64}, 2^{128}, 2^{256}, 2^{512}, and 2^{560}. What does this calculation tell you about whether or not 561 is prime? Can you use Fermat's test to determine whether or not 561 is prime?

3. The following proposition could be used while proving Fermat's Little Theorem:

Theorem. *p is a prime if and only if every $a \neq 0 \in \mathbb{Z}_p$ has a multiplicative inverse $1/a$ in $\mathbb{Z}_p$.*

Explain what this result means, and why it is true.

4. Give an inductive proof that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Remark: I'd really like an inductive proof, even if you know (for example, because we did this problem a while ago in class) some quicker way to prove this result.

5. Work out the sums

$$1 \cdot 2$$
$$1 \cdot 2 + 2 \cdot 3$$
$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4$$
$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5$$

and so on until you recognize the pattern (or remember what the formula is because we've talked about it before in class). Give a convincing inductive argument that the pattern continues. *Remark:* It wouldn't be that surprising if you ended up with a formula similar to the one in Problem 4, now would it?

6. Work out the sums

$$\frac{1}{1 \cdot 2}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5}$$

and so on until you recognize the pattern. Try to give a convincing inductive argument that the pattern continues.

7. Show that for every positive integer n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

You could do this inductively. You could also just use the results of Problems 4 and 5 and a bit of algebra.

∞. For extra credit, or just because you want to treat yourself to something fun and beautiful, try to generalize the results in Problems 4 and 5 (and, if you are feeling even cooler, Problem 6) as far as you can. Prove your conjectures inductively.