

4 Calculus A, Lab 4

At this stage in our quest to compute the derivative of every possible function, we can handle powers of x , and we know how to differentiate functions built up from powers of x using addition, subtraction, and multiplication by constants. That is, we know that if c is a constant, then

$$\frac{d}{dx}(cf(x)) = c \cdot f'(x).$$

We also know that

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

and that

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

if both the derivative on the right hand sides of these formulas exist.

The purpose of this lab is to see if we can find and defend similar formulas for

$$\frac{d}{dx}(f(x)g(x))$$

and for

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right).$$

1. A natural conjecture based on the formulas for the derivative of a sum or difference of 2 functions would be that

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g'(x).$$

Now, we already know how to compute the derivative of $k(x) = cg(x)$ by using the constant multiple formula. On the other hand, $k(x) = f(x)g(x)$ with $f(x) = c$. Is the known derivative $k'(x) = cg'(x)$ consistent with the conjecture that for every f and g , $\frac{d}{dx}(f(x)g(x)) = f'(x)g'(x)$?

2. If $f(x) = g(x) = x$, then we can compute the derivative of $f(x)g(x)$. Do so. Is the result we get equal to $f'(x)g'(x)$?

3. In order to seek the correct formula for the derivative of a product of 2 functions, fill in the blanks in the following table:

$f(x)$	$g(x)$	$k(x) = f(x)g(x)$	a	$f(a)$	$g(a)$	$f'(a)$	$g'(a)$	$k'(a)$
x	x		0					
$2x$	$3x$		0					
$2x$	$3x + 1$		0					
$2x$	$4x + 1$		0					
$3x$	$x + 5$		0					
$x + 1$	$4x$		0					
$5x + 1$	$4x$		0					
$x + 2$	$7x$		0					
$x + 1$	$x + 1$		0					
$x + 2$	$x + 1$		0					
$2x + 1$	$x + 1$		0					
$3x + 1$	$4x + 2$		0					
$x^2 + 3x + 1$	$x^2 - 2$		1					

Now conjecture a formula for the derivative $k'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$. Try more functions and more points if a pattern does not develop, or sit back and think. The pattern of zeros in the $f(a)$ and $g(a)$ columns is there to help you think.

A more difficult challenge: Can you think of a way to establish the truth of your conjecture in general?

4. To get a formula for the derivative of a quotient of 2 functions, let's start out easy by looking at reciprocals. That is, we'll let $r(x) = 1/g(x)$, and we'll try to compute $r'(x)$.

A natural conjecture would be that $r'(x) = 1/g'(x)$, i.e., that $\frac{d}{dx}(1/g(x)) = 1/g'(x)$. Try one or two functions at one or two points to test this conjecture. Linear functions, or even constant functions, should suffice.

5. In order to seek the correct formula for the derivative of $r(x) = 1/g(x)$, fill in the blanks in the following table. This is a bit more demanding than the previous table, since you can't yet compute the derivative $r'(a)$ analytically. Instead, you'll need either to zoom in to the graph with Maple and to try to estimate the slope, or to estimate

$$\lim_{h \rightarrow 0} \frac{r(a+h) - r(a)}{h}$$

numerically by using values of h close to 0.

$g(x)$	$r(x) = 1/g(x)$	a	$g(a)$	$g'(a)$	$r'(a)$
x		1			
$2x - 1$		1			
$3x - 2$		1			
$4x - 3$		1			
$x + 1$		1			
$2x$		1			
$3x - 1$		1			
$4x - 2$		1			
$x - 2$		1			
$2x - 1$		1			
$7x - 3$		1			
$x^2 + 3x + 1$		0			

Now conjecture a formula for the derivative $r'(x)$ in terms of $g(x)$ and $g'(x)$. Try more functions and more points if a pattern does not develop, or sit back and think. The pattern of values in the $g(a)$ column is there to help you think.

A more difficult challenge: Can you think of a way to establish the truth of your conjecture in general?

6. Combine your formulas for the derivatives of products and of reciprocals to get a formula for the derivative of the quotient of 2 functions. A hint:

$$\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$$

is a product of 2 functions, one of which is a reciprocal.