5 Calc A, Lab 5

5.1 Current Status

From class and from the last lab, we know that

$$\begin{aligned} (f(x) \pm g(x))' &= f'(x) \pm g'(x) \\ (f(x)g(x))' &= f(x)g'(x) + f'(x)g(x) \\ \left(\frac{f(x)}{g(x)}\right)' &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}. \end{aligned}$$

These rules, together with the rules that

$$\begin{aligned} \frac{d}{dx}(x^n) &= nx^{n-1} \\ \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x \end{aligned}$$

let us differentiate a large number of functions: anything built up from powers of x or from $\sin x$ and $\cos x$ by any combination of addition, subtraction, multiplication, and division.

There is one situation these rules do not yet cover, though. What if we build new functions by applying old functions one after another? That is, supposing we know the derivatives of $f(x)$ and $g(x)$, how would we find the derivative of $f(g(x))$?

5.2 Examples.

Examples of functions built up this way might be

(a) If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 2$, then $f(g(x)) = \sqrt{x^2 + 2}$.

(b) If $f(x) = x^2 - 3x + 4$ and $g(x) = 2x - 5$, then

$$f(g(x)) = (2x - 5)^2 - 3(2x - 5) + 4.$$

(c) If $f(x) = x^{100}$ and $g(x) = x^2 - 4x + 5$, then $f(g(x)) = (x^2 - 4x + 5)^{100}$.

(d) If $f(x) = \sin x$ and $g(x) = x^3$, then $f(g(x)) = \sin(x^3)$.

(e) If $f(x) = x^3$ and $g(x) = \sin x$, then $f(g(x)) = (\sin x)^3 = \sin^3 x$.

(f) If $f(x) = \sqrt{x}$ and $g(x) = 1 - \cos^2 x$, then $f(g(x)) = \sqrt{1 - \cos^2 x}$.

(g) On a fancier level, a function like

$$x + \sqrt{x + \sqrt{x + \sqrt{x}}}$$

is also built up by composition as well as by addition.

Some of these functions, like (b), (c) and (e) can be differentiated by first expanding them out and then using the rules we have; others we could handle only by going back to the definition involving limits. Wouldn't it be nice if we had a general formula for $\frac{d}{dx}(f(g(x)))$?

5.3 Problems.

1. A plausible conjecture would be that if $k(x) = f(g(x))$, then $k'(x) = f'(g(x))$. Try this rule out by computing $k'(x)$ and $f'(g(x))$ for the following pairs of functions:

(a) $f(x) = x^2$, $g(x) = 7x + 2$.
(b) $f(x) = x^2$, $g(x) = x^2$.
(c) $f(x) = x^2$, $g(x) = x^3$.
(d) $f(x) = x^2$, $g(x) = x^4$.
(e) $f(x) = x^2$, $g(x) = x^2 - 4x + 5$.
(f) $f(x) = x^2 - 3x + 4$, $g(x) = 2x - 5$.
(g) $f(x) = x$, $g(x) =$ a generic $g(x)$.

2. Based on what you learned in Problem 1, make a conjecture about how to compute the derivative of $k(x) = f(g(x))$. Test this conjecture by trying it out with $f(x) = \sqrt{x}$ and $g(x) = x^2+2$. That is, work out what your formula predicts the derivative to be, and then see whether your formula successfully calculates the slopes of tangent lines, or the numerical limits of difference quotients.

3. Once you have a working formula in Problem 2, use it to compute the derivatives of the functions (a)–(g) in the Examples section. If you're not sure of your results, try looking at tangent lines graphically or numerically to test your analytic results.

If you finish early, you could always work on Monday's homework.