8 Calculus A, Lab 8

8.1 Introduction.

This lab is intended to get us started on the other problem of calculus, the problem of computing the area under a curve. The basic task of the lab is to compute the area under a curve starting at a fixed point a and ending a various points x. This gives us a new function, the area function $F_a(x)$. We'll then compute the derivative of this area function, and see if we can understand what we observe.

I wanted us to do this lab using not just simple functions where we might be able to do everything algebraically, but also random functions where we had to work more geometrically. If our observations and conjectures hold for $f(x) = x^2$, it might be a coincidence, but if they work for a really complicated random function, then we gain much more confidence in them.

The complicated function I picked is the Riemann Zeta function, $\zeta(s)$. This turns out to be one of the most important functions in pure mathematics, but it's one you've probably never met. As a teaser, I've listed a few of the things every mathematician ought to know about the Zeta function in a appendix to this lab. For us right now, though, the only important thing is that we're working with a function that Maple calls Zeta(t). If you like, you might start out just by plotting $y = \zeta(t)$ on a few intervals, just to get a sense of what it looks like.

8.2 Area Functions.

Let $y = f(t)$ be any function. The area function $y = F_a(x)$ is defined to be the area above the t axis, below the graph of $y = f(t)$, to the right of the line $t = a$, and to the left of the line $t = x$. If one thinks of a as a constant, then this area is a function of x, which marks the right hand edge of the region whose area $F_a(x)$ measures. Figure 1 shows a sketch.

1. Let $f(t) = 2t + 3$, and let $a = 1$. The area function $y = F_a(x)$ is therefore the area above the t axis, below the graph of $y = 2t + 3$, to the right of the line $t = 1$, and to the left of the line $t = x$.

(a) Make a sketch showing the graph of $y = 2t + 3$ and the region whose area is $F_a(x)$.

(b) Compute the values of $F_a(2)$, $F_a(3)$, $F_a(\pi)$, $F_a(1)$, and $F_a(x)$.

(c) compute the derivative $F'_a(x)$. Do you notice anything?

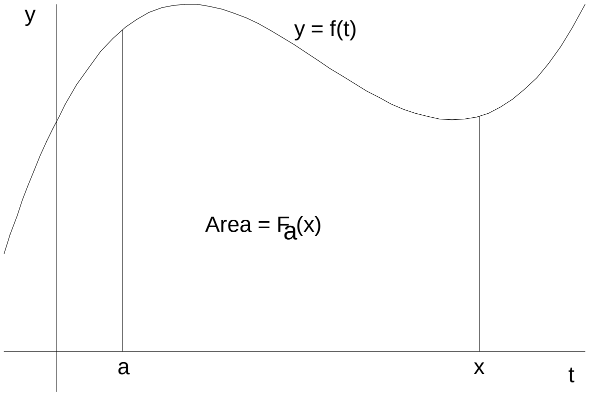

Figure 1: The area function $F_a(x)$.

2. Let $f(t) = \zeta(t)$ be the Riemann Zeta function, and let $a = -20$. Figure 2 is a plot of the Zeta function near a.

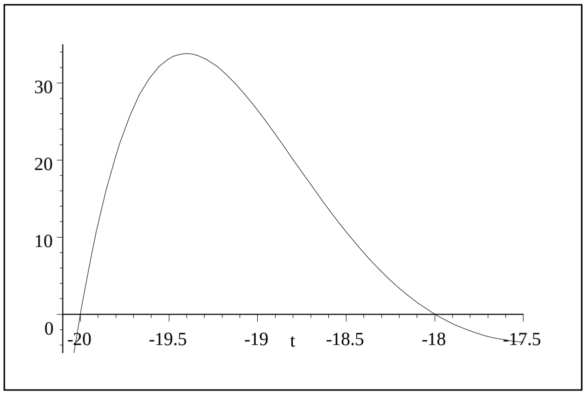

Figure 2: A bit of $y = \zeta(t)$.

Here is a table of some of the values (taken from Maple).

$\zeta(-20.0) = 0$
$\zeta(-19.9) = 11.707\,237\,1$
$\zeta(-19.8) = 20.561\,821$
$\zeta(-19.7) = 26.871\,821\,69$
$\zeta(-19.6) = 30.963\,861\,72$
$\zeta(-19.5) = 33.168\,325\,79$
$\zeta(-19.4) = 33.807\,683\,32$
$\zeta(-19.3) = 33.187\,616\,64$
$\zeta(-19.2) = 31.590\,624\,88$
$\zeta(-19.1) = 29.271\,766\,88$
$\zeta(-19.0) = 26.456\,212\,12$
$\zeta(-18.9) = 23.338\,283\,64$
$\zeta(-18.8) = 20.081\,698\,92$
$\zeta(-18.7) = 16.820\,741\,07$
$\zeta(-18.6) = 13.662\,121\,68$
$\zeta(-18.5) = 10.687\,327\,07$

(a) Estimate the value of $F_a(x)$ at each of the points listed above. Try to keep your total error less than about 1 if you can. I'd estimate the area by adding up a bunch of trapezoids.

(b) Estimate the value of $F_a'(x)$ at each of the points listed above. Try to keep your error less than about 1 if you can. I'd compute the derivatives by remembering that if h is small, then

$$F_a'(x) \doteq \frac{F_a(x+h) - F_a(x)}{h}.$$

Use the values for $F_a(x)$ from part (a) to do your calculations. Do you notice anything?

(c) Draw a sketch showing that as long as h is small, the numerator of the difference quotient

$$\frac{F_a(x+h) - F_a(x)}{h}$$

is very nearly the area of a trapezoid.

(d) Now try to get a very accurate estimate of $F_a'(-19)$ by using the formula for the area of a trapezoid to compute as accurately as you can the difference quotients

$$\frac{F_a(-19+h) - F_a(-19)}{h}$$

for $h = 0.1$, $h = 0.01$, $h = 0.001$, and $h = 0.0001$. (Remember that Maple can compute the Zeta function.) Does your value for $F_a'(-19)$ agree with what earlier problems have led you to conjecture?

3. The previous problems should have led you to make a conjecture about the relation between the function $f(t)$ and the derivative $F_a'(x)$ of the area function. State this conjecture as precisely as you can, and write a few sentences explaining why you think your conjecture is correct. You need not give a formal

algebraic argument—though I would be delighted if you did—but you should try to explain geometrically why your conjecture will work regardless of what your starting function f might be.

4. Suppose we had done Problem 1 starting with $f(t) = 2t + 3$ and with $a = -4$. Between $t = -4$ and $t = -2$, the graph of $f(t)$ lies beneath the t axis: $f(t) < 0$.

(a) How would you define the area function $F_a(x)$ in the region $-4 \leq x \leq -2$?

(b) Using your definition of $F_a(x)$, compute $F_a(-2)$ and $F_a'(-2)$.

(c) Is your conjecture working?

(d) If your conjecture fails in this case, then try to think of a new way to define $F_a'(x)$ which would preserve your conjecture. If your conjecture succeeds, then try to imagine other sensible definitions of $F_a(x)$, and see if your conjecture would work for them. In either case, what would you say to advocates of the competing definition?

8.3 Appendix: Fun Facts About the Riemann Zeta Function.

1. For real numbers $s > 1$, the Zeta function is given by

$$\begin{aligned}
\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ... + \frac{1}{n^s} + ... \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + ...\right) \cdot \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + ...\right) \cdot \\
&\quad \left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + ...\right) \cdot ... \\
&= \left(\frac{2^s}{2^s - 1}\right)\left(\frac{3^s}{3^s - 1}\right)\left(\frac{5^s}{5^s - 1}\right) ... \left(\frac{p^s}{p^s - 1}\right) ...
\end{aligned}$$

where the products run over all primes p. The infinite product representation means that the Zeta function contains a startling amount of information about the distribution of the prime numbers, which is why the Zeta function is so important.

For numbers which are not real numbers $s > 1$, $\zeta(s)$ does not have so convenient a representation.

2. For even positive integer values of s, the Zeta function has elegant expressions involving π. For instance,

$$\begin{aligned}
\zeta(2) &= \frac{\pi^2}{6}, \\
\zeta(4) &= \frac{\pi^4}{90}.
\end{aligned}$$

No such formulas are known for odd positive integer values of s; even the question of whether $\zeta(3)$ is a rational multiple of π^3 is open.

3. For negative even values of s,

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = ... = 0.$$

Every other root of the Zeta function is a complex number of the form $x + it$, where $i = \sqrt{-1}$, and where $0 \leq x \leq 1$. For infinitely many of these roots, $x = \frac{1}{2}$.

It is arguably true that the most important unsolved problem in mathematics is the Riemann Hypothesis, the conjecture that every non-trivial root of the Zeta function has the form $\frac{1}{2} + it$. There is a huge number of theorems in analytic number theory of the form, "If the Riemann Hypothesis is true, then..." Proving this conjecture would therefore settle in one stroke the truth of all these other results, resolving a very large family of questions in number theory.

David Hilbert, the greatest mathematician of the beginning of the last century, said that if he were to awake in 500 year like Barbarossa, the first question he would ask would be, "Has the Riemann Hypothesis been proved?" I'm less poetic, but if a genie were to grant me 3 wishes, it would be hard not to make one of them a request for a proof or disproof of the Riemann Hypothesis.