9 Calc A, Lab 9

9.1 Introduction.

We're pursuing two approaches to computing areas right now, one involving antiderivatives and one involving sums of areas of narrow slices. The first of these gives rise to the simplest formulas, but it is hampered because we can't always take the antiderivative of a function. As a result, when computing areas in real life, one is often reduced to adding up the areas of slices.

Unfortunately, we normally can't get exact formulas for the sum of the areas of a bunch of slices, since we don't have any great technology for computing sums. If we are reduced to adding up the areas of slices, then, we'll be just working numerically, not analytically.

The purpose of this lab is to look for ways to minimize our numerical work and to maximize the accuracy of our approximations if we are reduced to adding up slices.

9.2 Methods for approximating areas.

Suppose you want to compute the area under the graph of $y = f(x)$ between $x = 0$ and $x = 3$. Suppose, further, that you can't work out the result analytically, and that you need to compute approximate numerical areas of a bunch of slices and to add up the results. For concreteness, suppose you decide to cut the interval $[0, 3]$ into 4 slices. Then I can think of at least 4 methods you might use to approximate the areas.

9.2.1 Method 1: Right-hand rectangles.

In this method, which we've used in class, you would approximate f in each slice by a constant function whose height was the value of f at the right hand end of each slice. The area you're computing is then shown in Figure 1.

Formally, the area you're computing is

$$f\left(\frac{3}{4}\right)\frac{3}{4} + f\left(\frac{6}{4}\right)\frac{3}{4} + f\left(\frac{9}{4}\right)\frac{3}{4} + f\left(\frac{12}{4}\right)\frac{3}{4} = \sum_{k=1}^{4} f\left(\frac{3k}{4}\right)\frac{3}{4}.$$

9.2.2 Method 2: Left-hand rectangles.

Here you would use the same idea, except you would approximate f in each slice by a constant function whose height was the value of f at the left hand end of each slice. The area you're computing is then shown in Figure 2.

Formally, the area you're computing is

$$f\left(\frac{0}{4}\right)\frac{3}{4} + f\left(\frac{3}{4}\right)\frac{3}{4} + f\left(\frac{6}{4}\right)\frac{3}{4} + f\left(\frac{9}{4}\right)\frac{3}{4} = \sum_{k=0}^{3} f\left(\frac{3k}{4}\right)\frac{3}{4}.$$

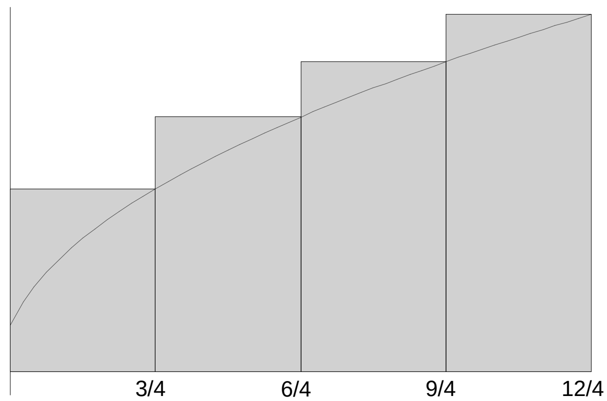

Figure 1: Right-hand rectangles.

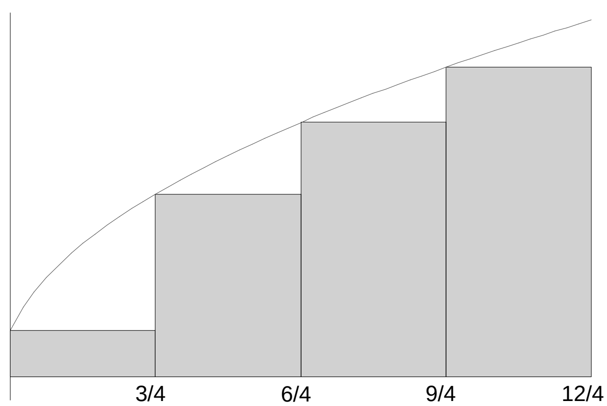

Figure 2: Left-hand rectangles.

9.2.3 Method 3: Trapezoids.

In lab last week, we approximated the areas by putting trapezoids in each slice. We could write down a new formula for this area, but there is a simpler way to think about it. In each slice, the area of the trapezoid is midway between the area of the left-hand rectangle and the area of the right-hand rectangle. The same thing has to work if we add up all the trapezoids: the area has to be the average of the areas of the right-hand and left-hand rectangles.

9.2.4 Method 4: Midpoints.

A final method would be to use rectangles again, but this time to let the height of the rectangle in each slice be the height of f at the midpoint of the slice. The area, shown in Figure 3, would be

$$f\left(\frac{0}{4}+\frac{3}{8}\right)\frac{3}{4}+f\left(\frac{3}{4}+\frac{3}{8}\right)\frac{3}{4}+f\left(\frac{6}{4}+\frac{3}{8}\right)\frac{3}{4}+f\left(\frac{9}{4}+\frac{3}{8}\right)\frac{3}{4}=\sum_{k=0}^{3} f\left(\frac{3k}{4}+\frac{3}{8}\right)\frac{3}{4}.$$

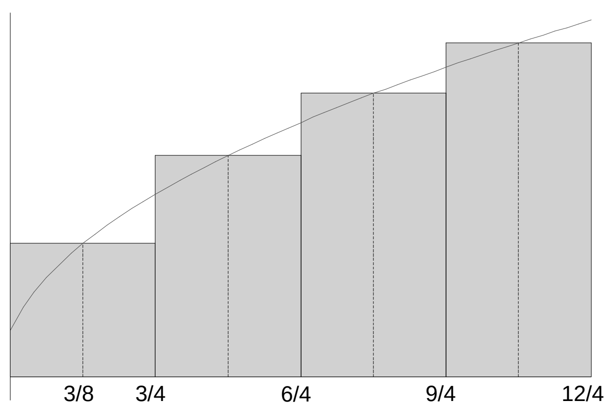

Figure 3: Midpoint rectangles.

9.2.5 General formulas.

In general, if we want to approximate

$$\int_a^b f(x)\,dx$$

using n slices, then the right-hand rectangle method would tell us to compute the sum

$$\sum_{k=1}^{n} f\left(a+\frac{(b-a)k}{n}\right)\frac{b-a}{n}.$$

The left-hand rectangle method says to compute

$$\sum_{k=0}^{n-1} f\left(a + \frac{(b-a)k}{n}\right)\frac{b-a}{n}.$$

The trapezoid method is the average of these two sums. The midpoint method says to compute

$$\sum_{k=0}^{n-1} f\left(a + \frac{(b-a)(k+\frac{1}{2})}{n}\right)\frac{b-a}{n}.$$

All these formulas are written in Maple form on the attached Maple worksheet.

9.3 Example.

Suppose you wanted to compute the area under the sine function between $x = 0$ and $x = \frac{\pi}{2}$,

$$\int_0^{\pi/2} \sin x \, dx.$$

We've never done a problem like this, so it looks like we'll have to approximate.

(Actually, the situation isn't hopeless. If we believe that the area function $F_0(x)$ satisfies $F_0'(x) = \sin x$ and that $F_0(0) = 0$, then $F_0(x) = 1 - \cos x$; so the area must be $F_0(\frac{\pi}{2}) = 1 - \cos\frac{\pi}{2} = 1$. This is useful to know as we start looking at the error in our approximations. But let's go back to assuming we need to approximate.)

1. Begin by filling in the following tables, showing the approximations produced by each method and showing their errors from the exact value of 1. Follow the advice on the Maple worksheet to avoid throwing away precision unnecessarily.

Approximation to the area

n	Right rects	Left rects	Trapezoids	Midpoints
1				
10				
100				
1000				
$10,000$				
$100,000$				
$1,000,000$				

Error in the approximation

n	Right rects	Left rects	Trapezoids	Midpoints
1				
10				
100				
1000				
$10,000$				
$100,000$				
$1,000,000$				

2. Some columns contain approximations that are too large (bigger than 1), and others contain approximations that are too small (less than 1). Explain geometrically how you could have determined in advance which columns would be which.

3. (a) Look at the errors in the right rectangle method. They should form a rough pattern. Are they proportional to $1/n$ or to $1/n^2$, or what? If you needed to work out the integral to 100 digit accuracy using the right rectangle method, roughly how big would n have to be? An order of magnitude estimate is fine. Don't try actually getting that level of accuracy in the computation; you might not like the wait.

(b) Repeat part (a) for the left rectangle method.

(c) Repeat part (a) for the trapezoid method.

(d) Repeat part (a) for the midpoint method.

4. Compare the errors of the midpoint and trapezoid methods. Which worked better? Is there any relation between the sizes and signs of the errors in the two methods?

5. Exploit the connection in Problem 4 to suggest a way to combine the midpoint and the trapezoid method into a new method more accurate than either alone. Try out the new method on our sample integral, and repeat problem 3(a) with the new method. Don't quit until you have a method which is way better than the midpoint method. Just twice as good is not enough.

6. Once you have a really good method in Problem 5, try it out on a quadratic function. See if you can prove anything.