1 Introduction and Review

1.1 A First Glimpse of Calculus

In high school, I didn't study calculus, but I remember reading about it in Encyclopaedia Britannica. While I didn't understand a lot of what I read, I remember discerning 3 general facts:

1. Calculus involves computing the slope of curves.
2. Calculus involves computing the areas under curves.
3. Calculus involves notions from physics like velocity and acceleration.

At first glance, these concepts seem completely unrelated. Shouldn't Calculus really be 2 or 3 separate subjects? The answer turns out to be no. Calculus is really one subject, and one of the central features of that subject is the deep interconnectedness of the 3 ideas of slope, area, and dynamical change. Exploring this connection is one of the main objectives of this course, but let me start by trying to show you where we're heading.

Let's start with the simplest possible curve, a horizontal straight line, say, $y = 65$.

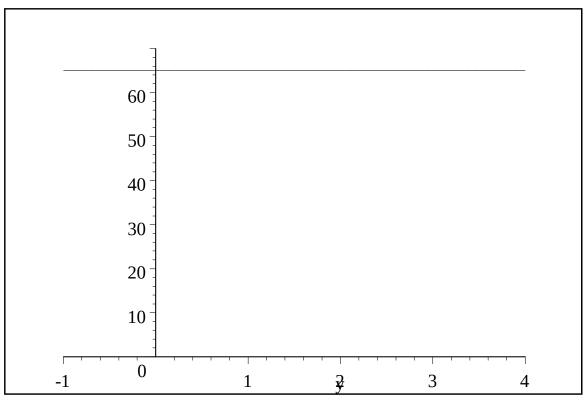

Figure 1: $y = 65$.

Suppose we want to compute the area under this graph. We can't mean the total area between $t = -\infty$ and $t = \infty$, since that would be infinite. Instead, we must mean the area under the graph and above the t-axis between, say, $t = 0$ and $t = x$. For every choice of x we get a different area; the area being the area of a rectangle with height 65 and width x. In other words, $A = 65x$.

Now this area can be regarded as a function in its own right, and plotted like Figure 2.

At this stage, we've solved the problem of finding the area under our very simple curve, one of the 3 problems I said calculus was about. Now here comes the nice discovery. Suppose we were to look at the second function, the one plotted in Figure 2, and to ask for its slope. What would we get? The slope is

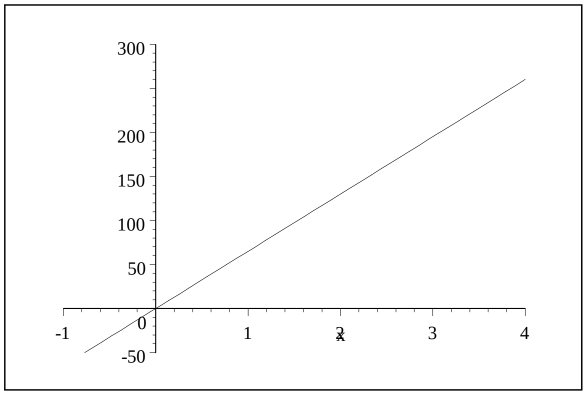

Figure 2: $A = 65x$.

65, and the slope doesn't change. It's always the same at every point. So if we were to plot the slope, we'd get a constant function $y = 65$ shown in Figure 3.

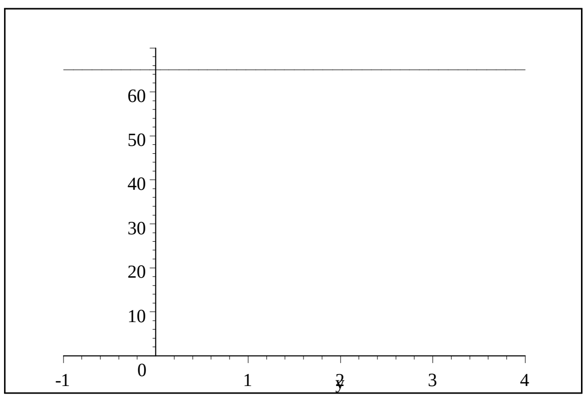

Figure 3: $y = 65$.

Have you ever seen a plot like this before? Of course: it's the original function in Figure 1! So what we seem to see in this example is that the problem of computing the slope of a function and the problem of computing the area under a graph aren't 2 different problems - they're the same problem done in different directions. The area problem leads us from Figure 1 to Figure 2, and the slope problem leads us from Figure 2 back to Figure 1.

This isn't just a coincidence of picking a very particular function to begin with. It works in general, and it is one way of stating a very important result called the Fundamental Theorem of Calculus.

So now we have a connection between the problem of slopes and the problem of areas. What's the connection to physics? Well, suppose the original function $y = 65$ represented a velocity—someone driving at a constant 65 mi/ h. What does the area function $y = 65t$ represent? It's the total distance travelled, starting at time $t = 0$. Put another way, it's the position of the car if its starting position is 0. So when we take the slope of the position function, we get the velocity function, and when we take the area under the velocity function, we get the position function. Since in physics, one is often moving back and forth between position and velocity, calculus begins to feel like a useful tool for doing physics, too.

1.1.1 A second example

What if we were to start with a more complicated function than a horizontal straight line? The next simplest function would be a slanted line through the origin, like $y = 2t$, which is shown in Figure 4.

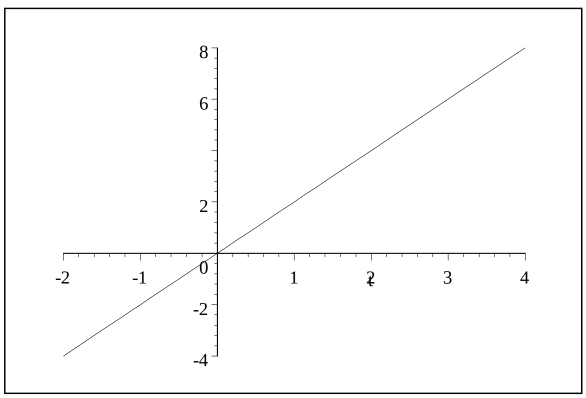

Figure 4: $y = 2t$.

The area under this curve between $t = 0$ and $t = x$ is the area of a triangle with base x and height $2x$. This area is $A = x^2$, which is plotted as Figure 5.

Now things get tricky. We think that if we start with $y = 2t$, take the area to get

$A = x^2$, and then take the slope of this new curve, we will get back to where we started. That is, the slope of the parabola $A = x^2$ clearly differs from point to point, but we think that at any particular point $x = t$, the slope of the curve $A = x^2$ will turn out to be $2t$. Is this reasonable?

Qualitatively, it is certainly plausible. The slope of the parabola at $x = t$ that is positive for positive t, negative for negative t, and 0 for $t = 0$. The value of $y = 2t$ (its height above the x-axis) is also positive, negative, or 0 according as t is positive, negative, or 0. Further, it is obvious from symmetry that the slope of the parabola at $x = -t$ is the negative of the slope at $x = t$; the height of $y = 2t$ has this same symmetry. So the slope of $A = x^2$ at $x = t$ is given

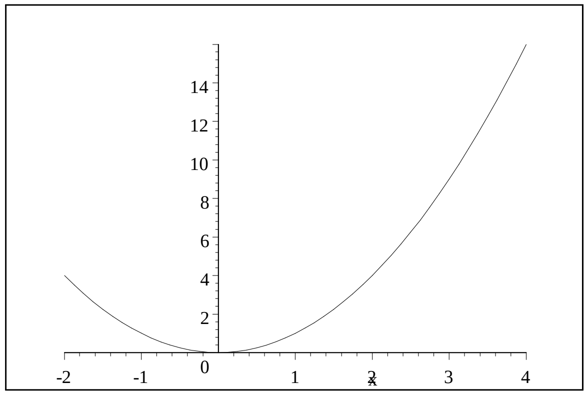

Figure 5: $A = x^2$.

either by $y = 2t$ or by some very closely related function.

To get a quantitative estimate of these slopes, we should probably pick a particular choice of t, say, $t = 1$. Figure 6 shows a picture of the parabola near $t = 1$.

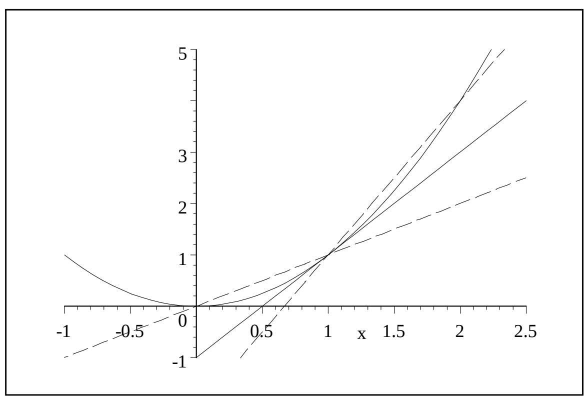

Figure 6: A parabola.

With this parabola, I've drawn 3 lines. The solid curve is the tangent line, whose slope we don't know how to compute. The 2 dashed lines are secant lines, lines that intersect the curve at 2 points. We can easily compute their slopes.

The steeper dashed line intersects the parabola at the points $(1, 1)$ and $(2, 4)$. Its slope is therefore $\frac{4-1}{2-1} = 3$. It is steeper than the tangent line.

The shallower dashed line intersects the parabola at the points $(0, 0)$ and $(1, 1)$. Its slope is therefore $\frac{1-0}{1-0} = 1$. It is not as steep as the tangent line.

The slope of the tangent line must lie somewhere between the slopes of the

2 dashed secant lines. Somewhere between 1 and 3 might be, maybe, 2? And in fact, this is the slope we were expecting, since our conjecture has been that the slope of the tangent line to $A = x^2$ at the point $x = t$ that is $2t$.

It is worth reflecting on how we could get a better approximation to the slope of the tangent line. It is also worth trying the same calculation at points other than $x = 1$.

1.2 Basic Algebra Review

This is a quick reminder of some of the basics of algebra you might find yourself needing to know. I make no claim of comprehensiveness.

The Pythagorean Theorem implies that the distance between the points (x_0, y_o) and (x, y) is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

The slope of a straight line is $\Delta y/\Delta x$, where Δy is the difference in the y coordinates of 2 points on the line, and Δx is the difference in their x coordinates.

The straight line with slope m and y-intercept b is $y = mx + b$.

The line through the point (x_0, y_0) and having slope m is

$$\frac{y - y_0}{x - x_0} = m,$$

i.e., $y - y_0 = m(x - x_0)$.

The line through the two points (x_0, y_0) and (x_1, y_1) has slope

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}.$$

Its equation is therefore

$$y - y_0 = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0).$$

Example: The line through the points $(1, 3)$ and $(5, 14)$ has equation

$$y - 3 = \frac{14 - 3}{5 - 1}(x - 1).$$

The circle of radius r centered at (x_0, y_0) consists of all points (x, y) whose distance from (x_0, y_0) is r. This means the circle has equation

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r,$$

or, more simply,

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

A circle of radius r has area πr^2 and circumference $2\pi r$.

The roots of the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Every one of these facts is worth memorizing.

1.3 Trigonometry Review

The trig functions are functions that take angles as arguments, and return numbers as values.

In calculus, as throughout mathematics, angles are almost always described in radians. The radian measure of an angle is the length of the arc on the unit circle contained within the angle, as shown in Figure 7.

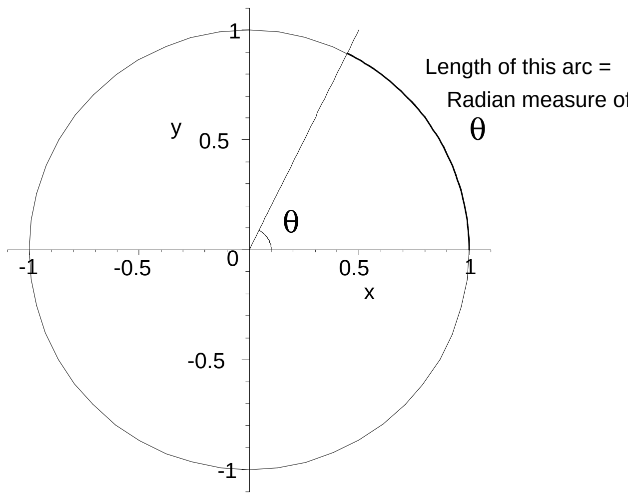

Figure 7: Radian measure.

The circumference of the unit circle is 2π, so a 360° angle measures 2π radians. Other radian measures people often remember are

$$\begin{aligned} 180^\circ &= \pi\,\text{rad} \\ 90^\circ &= \frac{\pi}{2}\,\text{rad} \\ 45^\circ &= \frac{\pi}{4}\,\text{rad} \\ 30^\circ &= \frac{\pi}{6}\,\text{rad} \\ 60^\circ &= \frac{\pi}{3}\,\text{rad} \\ 120^\circ &= \frac{2\pi}{3}\,\text{rad} \end{aligned}$$

To convert from radians to degrees, you therefore multiply by $180/\pi$. To convert from degrees to radians, multiply by $\pi/180$. After all, the units cancel nicely in a calculation like

$$37^\circ = 37^\circ \times \frac{\pi\,\text{rad}}{180^\circ} = 0.645\,77\,\text{rad}.$$

The 2 basic trig functions, cosine and sine, are defined as the x and y coordinates, respectively, of points on the unit circle. That is, the point (x, y) shown in Figure 8 on the unit circle at an angle θ from the x axis has coordinates $x = \cos\theta$, $y = \sin\theta$. It could have been written as $(x, y) = (\cos\theta, \sin\theta)$.

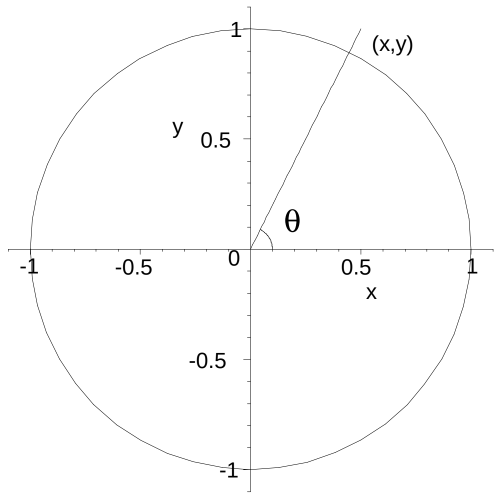

Figure 8: $(x, y) = (\cos\theta, \sin\theta)$.

To plot the trig functions as functions of θ, one wants the horizontal axis to be θ, and one wants the vertical axis to show the value of the trig function at angle θ. If you walk around the unit circle starting on the x axis at $\theta = 0$, then you find that

When $\theta = 0$, the x coordinate of the point on the unit circle is 1; so $\cos 0 = 1$.

When $\theta = \frac{\pi}{2} = 90\,^\circ$, the x coordinate of the point on the unit circle is 0; so $\cos\frac{\pi}{2} = 0$.

When $\theta = \pi = 180\,^\circ$, the x coordinate of the point on the unit circle is -1; so $\cos\pi = -1$.

When $\theta = \frac{3\pi}{2} = 270\,^\circ$, the x coordinate of the point on the unit circle is 0; so $\cos\frac{3\pi}{2} = 0$.

When $\theta = 2\pi = 360\,^\circ$, the x coordinate of the point on the unit circle is 1; so $\cos 2\pi = 1$. At this point, we are back to our starting point on the circle; for larger values of θ, the values of the cosine function will just repeat.

At this stage, we know that the points $(0, 1)$, $(\frac{\pi}{2}, 0)$, $(\pi, -1)$, $(\frac{3\pi}{2}, 0)$, $(2\pi, 1)$ are on the graph of the cosine function. The full graph looks something like Figure 9.

Doing the same thing with $\sin\theta$ produces the graph in Figure 10.

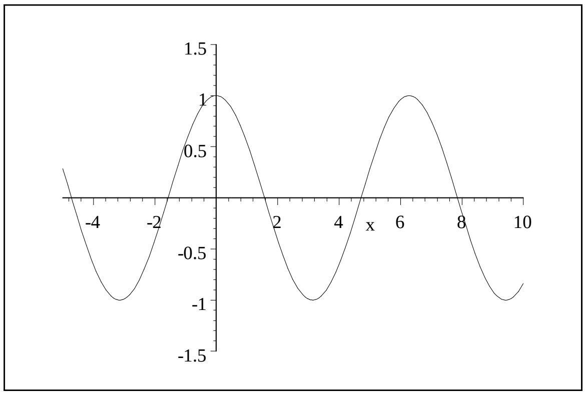

Figure 9: $y = \cos\theta$.

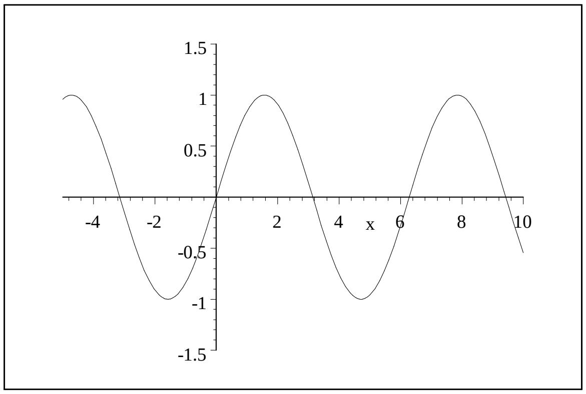

Figure 10: $y = \sin\theta$.

1.3.1 Two Remarks on Notation

Standard math notation is more ambiguous and context-sensitive than some people realize. Sine and cosine are functions, so in the same way that we write $f(x)$, we really ought to write $\sin(\theta)$ and $\cos(\theta)$. That way, it would look like we were applying a function, not multiplying quantities. Maple requires us to use this notation, but outside Maple, it is universal practice to leave out the parentheses. It is also misleading.

The notation $\sin^2\theta$ is even more confusing. Does it mean $\sin(\sin(\theta))$ or what? No, in fact, it means $(\sin(\theta))^2$. That is, it means to start with the angle θ, take the sine to get a number, and then square this number. Again, Maple mandates the notation $(\sin(\theta))^2$ rather than the shorthand $\sin^2\theta$. I'll use the universal shorthand, but you should be sure you understand what it means.

1.3.2 Basic Trig Identities

The unit circle consists of all points that are a distance 1 from the origin, i.e., of all points (x, y) satisfying $\sqrt{x^2 + y^2} = 1$. Squaring both sides of this equation gives a simpler equation for the circle: $x^2 + y^2 = 1$. Since the point on the unit circle at angle θ has $x = \cos\theta$ and $y = \sin\theta$, this equation immediately implies the most well-known of the trig identities, the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1.$$

Two more identities are obvious either from looking at the graphs in Figures 9 and 10 or from thinking about the unit circle below in Figure 11.

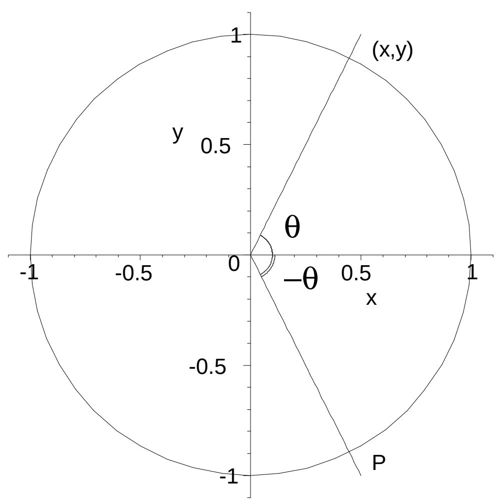

Figure 11: θ and $-\theta$.

The point P in this figure could be written either as $(\cos(-\theta), \sin(-\theta))$ or as $(x, -y) = (\cos\theta, -\sin\theta)$. This gives us the identities

$$\begin{aligned} \cos(-\theta) &= \cos(\theta) \\ \sin(-\theta) &= -\sin(\theta). \end{aligned}$$

These identities offer an opportunity to introduce a pair of general concepts. A function f for which $f(-x) = f(x)$ for all x is called an *even* function. A function g for which $g(-x) = -g(x)$ for all x is called an *odd* function. The identities above can be stated simply by saying that cosine is an even function, and that sine is an odd function.

Even functions are symmetric across the y-axis; the left and right sides are identical. Figure 12 shows a typical even function.

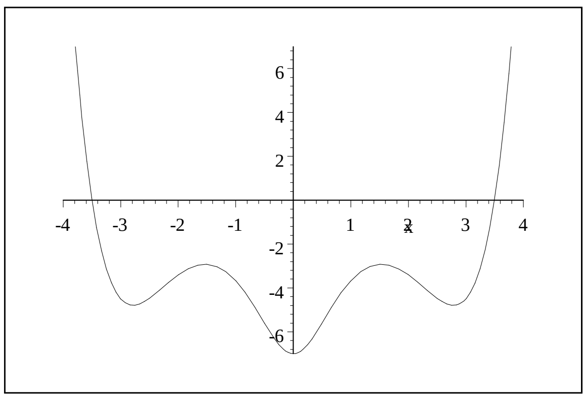

Figure 12: An even function.

Odd functions are symmetric about the origin. That is, the graph of an odd function is unchanged if you stick a pin in the origin and rotate the page 180° in the plane of the paper. A typical odd function is shown in Figure 13.

A last basic identity could be seen by looking at the graphs of the sine and cosine functions in Figures 9 and 10. The two graphs have the same shape, except that the graph of the cosine function seems to be the graph of the sine function slid left by a distance $\pi/2$. We saw in the first lab that to slide a graph left by a distance k you add k to the value of the argument. Thus,

$$\cos\theta = \sin(\theta + \frac{\pi}{2})$$

or, equivalently,

$$\sin\theta = \cos(\theta - \frac{\pi}{2}).$$

To prove these identities more compellingly from the unit circle, think about the picture in Figure 14, which shows the angle θ, the angle $\theta + \frac{\pi}{2}$, and the points these angles cross the unit circle. It is clear geometrically that the y coordinate

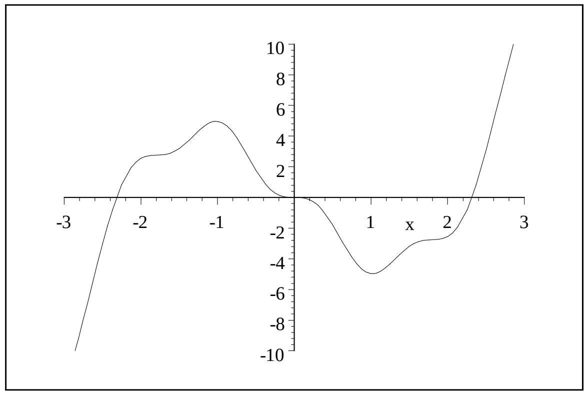

Figure 13: An odd function.

of point at which the larger angle meets the unit circle is the same as the x coordinate of the point at which the smaller angle meets the unit circle, i.e., that $\sin(\theta + \frac{\pi}{2}) = \cos\theta$.

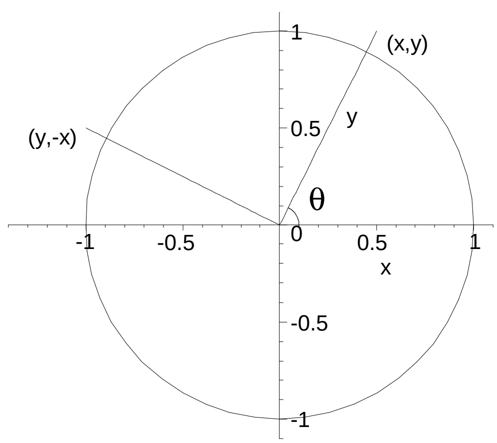

Figure 14: θ and $\theta + \frac{\pi}{2}$.

For completeness, let me list 2 more identities in the same family as these:

$$\begin{aligned} -\cos\theta &= \sin(\theta - \frac{\pi}{2}), \\ -\sin\theta &= \cos(\theta + \frac{\pi}{2}). \end{aligned}$$

These are easy consequences of the identities we already have.

1.3.3 Other Trig Functions

In addition to the sine and cosine, there are 4 other trig functions, the secant, cosecant, tangent, and cotangent, which can be defined as

$$\begin{aligned}
\sec\theta &= \frac{1}{\cos\theta} \\
\csc\theta &= \frac{1}{\sin\theta} \\
\tan\theta &= \frac{\sin\theta}{\cos\theta} \\
\cot\theta &= \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

The graphs of the secant and cosecant functions can easily be obtained from the graphs of sine and cosine. Since $\sec\theta = \frac{1}{\cos\theta}$, we know that when $\cos\theta$ is slightly less than 1, $\sec\theta$ will be slightly more than 1. When $\cos\theta$ is small and positive, $\sec\theta$ will be large and positive. Similarly, when $\cos\theta$ is slightly bigger than -1, $\sec\theta$ will be slightly less than -1, and when $\cos\theta$ is small and negative, $\sec\theta$ will be large and negative. Graphs of $y = \cos\theta$ and $y = \sec\theta$ are plotted together in Figure 15, and graphs of $y = \sin\theta$ and $y = \csc\theta$ are plotted together in Figure 16. In these plots, the sine and cosine functions are drawn as dashed lines, and the secant and cosecant are drawn as solid lines.

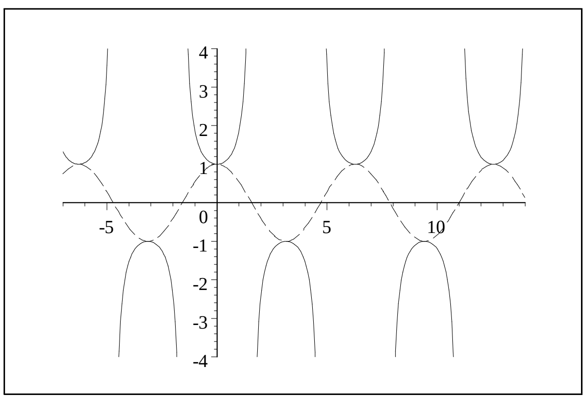

Figure 15: $y = \sec\theta$.

The functions $\sec\theta$ and $\csc\theta$ are less important than $\sin\theta$ and $\cos\theta$, since it is easier to think of physical things that look like waves than it is to think of physical things that shoot off to ∞, reappear at $-\infty$, zoom up toward the y axis only to rush back off to $-\infty$, and then repeat this again and again.

To graph the tangent function, and to see its importance, it's worthwhile to go back to the unit circle. The point in Figure 17 on the unit circle at an angle θ from the x axis has coordinates $y = \sin\theta$, $x = \cos\theta$. Thus, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$ is the slope of the line between this point and the origin.

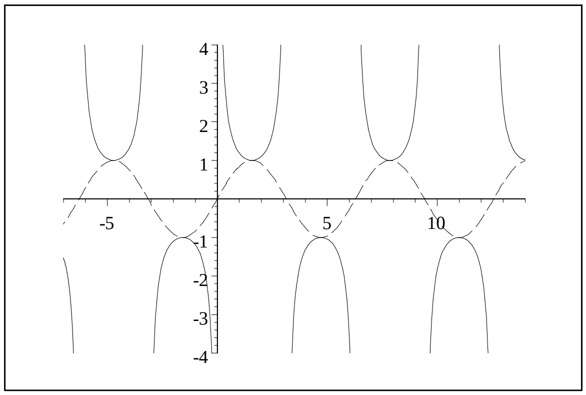

Figure 16: $y = \csc\theta$.

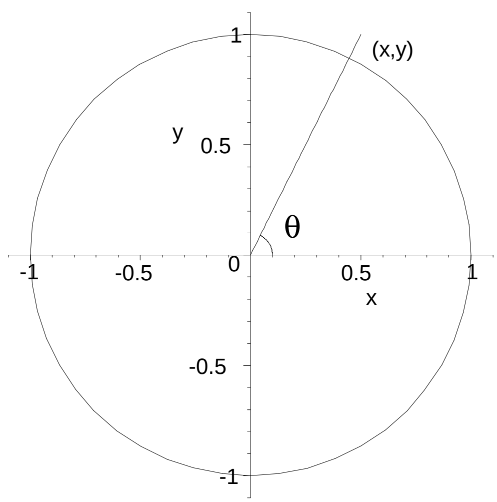

Figure 17: The unit circle again.

It's now easy to sketch a graph of $y = \tan\theta$. When $\theta = 0$, the slope of the line is 0, so $\tan 0 = 0$. As θ grows, the slope increases, until as θ approaches $\frac{\pi}{2} = 90°$, the slope approaches ∞. Starting from 0 and decreasing, the slope becomes more and more negative, approaching $-\infty$ as θ gets close to $-\frac{\pi}{2} = -90°$. If θ is slightly larger than $\frac{\theta}{2}$, then the slope of the line is large and negative. It increases to 0 when $\theta = \pi$, and continues to increase to approach ∞ as θ gets close to $\frac{3\pi}{2}$. Figure 18 shows a sketch of $y = \tan\theta$.

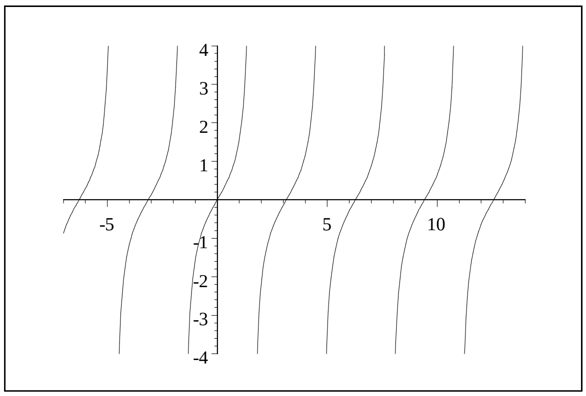

Figure 18: $y = \tan\theta$.

At a first glance, $\tan\theta$ looks something like x^3 near the origin, but there are subtle differences. $y = x^3$ has a slope of 0 at $x = 0$, while $y = \tan\theta$ has a slope of 1.

It's not hard to play the same game to get a graph of the cotangent, shown in Figure 19.

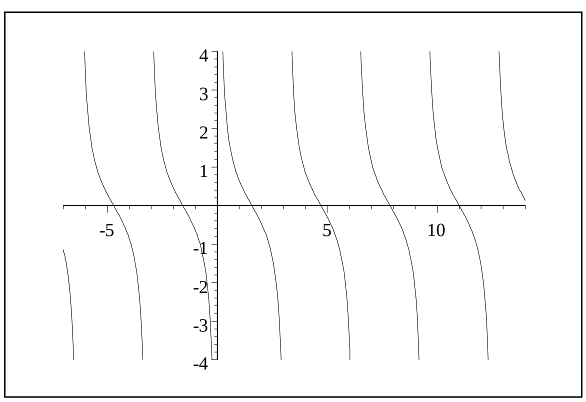

Figure 19: $y = \cot\theta$.

1.3.4 Periodicity

We've noticed that $\sin\theta$ and $\cos\theta$ repeat after $2\pi = 360\,^\circ$. The same can easily be seen to hold for $\sec\theta$ and $\csc\theta$. On the other hand, a bit of thought or a glance at the plots shows that $\tan\theta$ and $\cot\theta$ repeat after on $\pi = 180\,^\circ$. Algebraically, for every integer n,

$$\begin{aligned}
\sin(\theta + 2\pi n) &= \sin\theta \\
\cos(\theta + 2\pi n) &= \cos\theta \\
\sec(\theta + 2\pi n) &= \sec\theta \\
\cot(\theta + 2\pi n) &= \cos\theta \\
\tan(\theta + \pi n) &= \tan\theta \\
\cot(\theta + \pi n) &= \cot\theta
\end{aligned}$$

1.3.5 Basic identities for the other trig functions

It's easy to see looking at the graphs that $\sec\theta$ and $\cot\theta$ are even functions, and that $\csc\theta$ and $\tan\theta$ are odd functions.

$$\begin{aligned}
\sec(-\theta) &= \sec\theta \\
\cot(-\theta) &= \cot\theta \\
\tan(-\theta) &= -\tan\theta \\
\csc(-\theta) &= -\csc\theta
\end{aligned}$$

There is also a useful identity for secant and tangent that resembles the Pythagorean identity, and that is obtained from it.

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
\tan^2\theta + 1 &= \sec^2\theta
\end{aligned}$$

Personally, I can never remember whether the identity is $\tan^2\theta - \sec^2\theta = 1$ (wrong) or whether it's $\sec^2\theta - \tan^2\theta = 1$ (right); so I just remember how to prove the identity when I need it.

1.3.6 Connections with triangles

Why are there exactly 6 trig functions? What motivated the odd definitions of the secant and tangent and so on? And what does trigonometry have to do with triangles? ("Gnomon" means "angle" in Greek.) To answer this question, think about an arbitrary right triangle. This triangle can be related to the unit circle as shown in Figure 20.

The 2 triangles in this figure are similar. The smaller triangle has edges $\cos\theta$, $\sin\theta$, and 1. The corresponding sides of the large triangle are traditionally

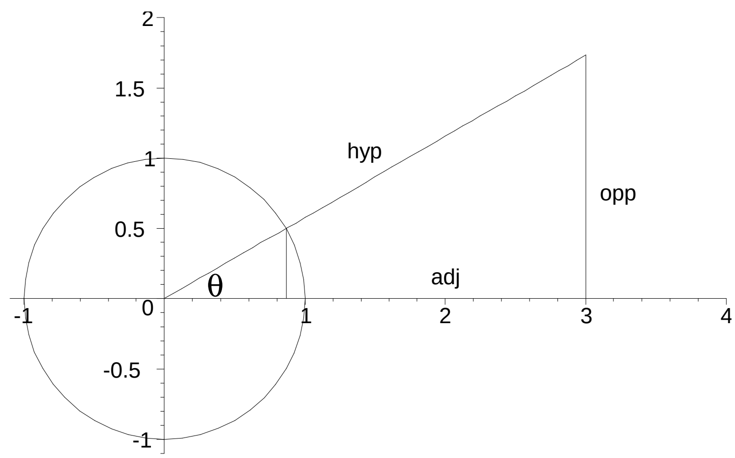

Figure 20: Trig functions and triangles.

called the adjacent and opposite sides, and the hypotenuse. Thinking about similar triangles now gives us

$$\begin{aligned}
\sin\theta &= \frac{\sin\theta}{1} = \frac{opp}{hyp} \\
\cos\theta &= \frac{\cos\theta}{1} = \frac{adj}{hyp} \\
\tan\theta &= \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj} \\
\sec\theta &= \frac{1}{\cos\theta} = \frac{hyp}{adj} \\
\cot\theta &= \frac{\cos\theta}{\sin\theta} = \frac{adj}{opp} \\
\csc\theta &= \frac{1}{\sin\theta} = \frac{hyp}{opp}
\end{aligned}$$

The trig functions are therefore the 6 possible ways to select 2 different sides of a right triangle and to take their ratio.

1.3.7 Sum, Double and Half Angle Formulas

In high school, you may have seen scores of identities connecting the trig functions. Two important ones we'll make reference to once or twice are the sum formulas,

$$\begin{aligned}
\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
\sin(\theta+\phi) &= \cos\theta\sin\phi + \sin\theta\cos\phi
\end{aligned}$$

If we set $\phi = \theta$, then these become the double angle formulas,

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta\end{aligned}$$

The first of the double angle formulas can be transformed to give

$$\cos(2\theta) = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1,$$

which can be solved to give

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}.$$

Alternatively, one could write

$$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta,$$

which solves to give

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}.$$

For completeness (not for any imaginable test), let me show you where the sum formulas come from. Figure 21 is a drawing of the unit circle, with 4 points shown. Point X is the point $(1, 0)$ at which the x axis meets the unit circle. Point B is an angle θ above X. Point A is an angle ϕ above B, and therefore an angle $\theta + \phi$ above X. Point C is an angle ϕ below X, and therefore an angle $\theta + \phi$ below point B.

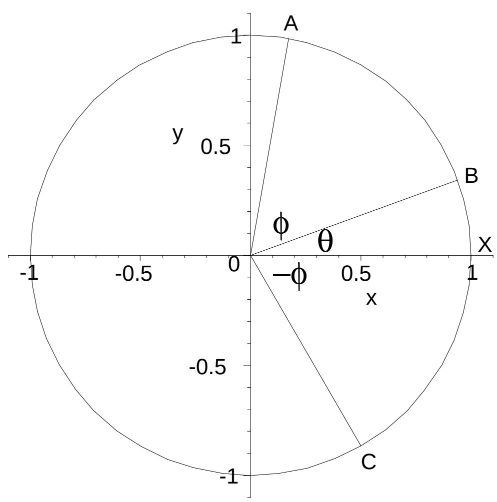

Figure 21: The sum formula.

The coordinates of the points in the Figure are therefore

$$\begin{aligned} X &= (0,0) \\ B &= (\cos\theta, \sin\theta) \\ A &= (\cos(\theta+\phi), \sin(\theta+\phi)) \\ C &= (\cos(-\phi), \sin(-\phi)) = (\cos\phi, -\sin\phi) \end{aligned}$$

Now, the points A and X are the same distance apart as the points B and C, since they are separated by the same angles on the unit circle. The Pythagorean Theorem tells us that these distances are

$$\begin{aligned} \overline{AX}^2 &= [\cos(\theta+\phi) - 1]^2 + [\sin(\theta+\phi) - 0]^2 \\ &= \cos^2(\theta+\phi) - 2\cos(\theta+\phi) + 1 + \sin^2(\theta+\phi) \\ &= 2 - 2\cos(\theta+\phi) \end{aligned}$$

and

$$\begin{aligned} \overline{BC}^2 &= [\cos\theta - \cos\phi]^2 + [\sin\theta + \sin\phi]^2 \\ &= \cos^2\theta - 2\cos\theta\cos\phi + \cos^2\phi + \sin^2\theta + 2\sin\theta\sin\phi + \sin^2\phi \\ &= 2 - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi. \end{aligned}$$

Setting these 2 expressions equal gives

$$\begin{aligned} -2\cos(\theta+\phi) &= -2\cos\theta\cos\phi + 2\sin\theta\sin\phi \\ \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \end{aligned}$$

which is the first of the sum formulas. To get the second sum formula, use apply the identity $\sin\alpha = \cos(\frac{\pi}{2} - \alpha)$ to the first sum formula. You get

$$\begin{aligned} \sin(\theta+\phi) &= \cos\left(\frac{\pi}{2} - (\theta+\phi)\right) \\ &= \cos\left(\frac{\pi}{2} - \theta\right)\cos(\phi) - \sin\left(\frac{\pi}{2} - \theta\right)\sin\phi \\ &= \sin\theta\cos\phi + \sin\theta\cos\phi. \end{aligned}$$

1.3.8 Special Values

For a few values of θ, we can easily compute $\sin\theta$ and $\cos\theta$ exactly. If $\theta = \frac{\pi}{4} = 45\,^\circ$, then the point (x, y) lies both on the unit circle $x^2 + y^2 = 1$ and on the line $y = x$. This means that $x^2 + x^2 = 1$, so $x^2 = \frac{1}{2}$, so $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Since $y = x$, $y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ as well.

If $\theta = \frac{\pi}{3} = 60\,^\circ$, then it is not hard to see geometrically that there is an equilateral triangle present in our usual figure, as shown in Figure 22.

It is clear from the figure that $x = \cos\frac{\pi}{3} = \frac{1}{2}$. We can then solve $x^2 + y^2 = 1$ (the equation of the unit circle) to get $y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. A similar construction lets one compute the sine and cosine of $\frac{\pi}{6} = 30\,^\circ$, or you can get them from the identities we've already derived. The results are summarized in the table below.

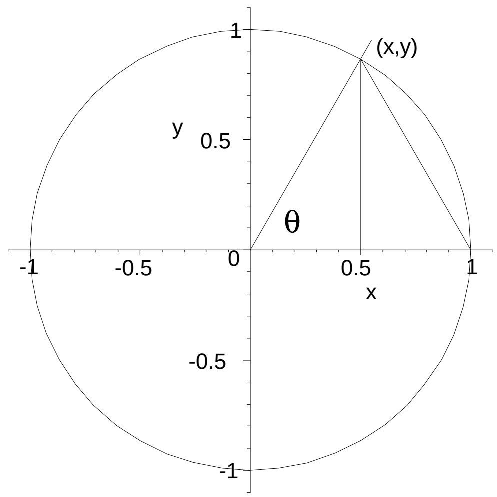

Figure 22: $\theta = \frac{\pi}{3}$.

θ	$\cos\theta$	$\sin\theta$
0	1	0
$\pi/6 = 30^\circ$	$\sqrt{3}/2$	$1/2$
$\pi/4 = 45^\circ$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/3 = 60^\circ$	$1/2$	$\sqrt{3}/2$
$\pi/2 = 90^\circ$	0	1
$2\pi/3 = 120^\circ$	$-1/2$	$\sqrt{3}/2$