2 Limits

2.1 What are limits, and why do we care?

Here's a preview to the solution to one of the central problems in calculus, the problem of computing the slope of a curve. If we wanted to approximate the slope of the tangent line to the curve $y = x^2$ at the point

$x = 1$, $y = 1$, we could take the point $(1, 1)$ along with a second point on the curve, join these points with a straight line, and compute the slope of this secant line. For instance, if the second point were $(2, 4)$, then the slope of the secant line would be $\frac{4-1}{2-1} = 3$. If the second point were $(0, 0)$, then the slope of the secant line would be $\frac{0-1}{0-1} = 1$. Both these lines are shown in Figure 1, where it is clear that the first is too steep to be the tangent line, and that the second is not steep enough. The slope of the tangent line must therefore be between 1 and 3.

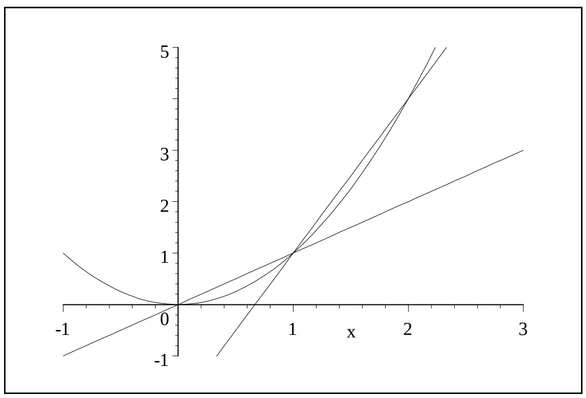

Figure 1: Two secant lines.

To get a better approximation to the slope of the tangent line, one ought to pick the second point on the secant line to be closer to $(1, 1)$. For example, if the second point were $(1.5, 2.25)$, then the slope of the secant line would be $\frac{2.25-1}{1.5-1} = 2.5$. If the second point were $(0.5, 0.25)$, then the slope of the secant line would be $\frac{0.25-1}{0.5-1} = 1.5$. Again, as shown in Figure 2, the first of these is too steep to be the tangent line, and the second is too shallow. So the slope of the tangent line must be somewhere between 1.5 and 2.5.

To get a really good approximation to the slope of the tangent line, we ought to pick the second point to be very close to $(1, 1)$, since it seems to make sense that the tangent line, which meets the curve at only one point, should be well approximated by secant lines intersecting the curve at 2 points very close together. So we should take the second point not at $x = 2$ or at $x = 1.5$, but at $x = 1.1$ or at $x = 1.01$ or at $x = 1.001$. That is, we should approximate the tangent line with the line through the point $(1, 1)$ and a point like $(1.1, 1.1^2)$ or

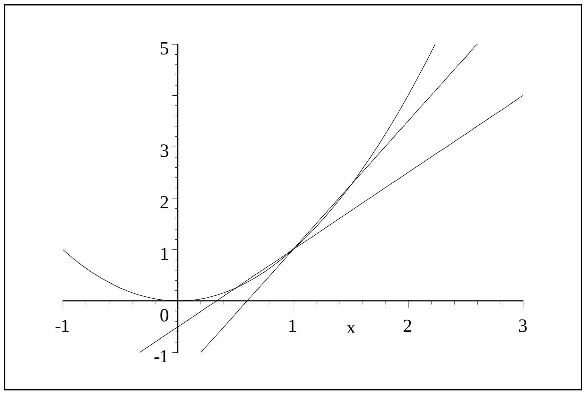

Figure 2: Secant lines closer to the tangent.

$(1.01, 1.01^2)$ or $(1.01, 1.01^2)$. The slopes of these secant lines are listed in the following table.

x	$(x, f(x))$	slope
1.1	$(1.1, 1.21)$	$\frac{1.21-1}{1.1-1} = 2.1$
1.01	$(1.01, 1.0201)$	$\frac{1.0201-1}{1.01-1} = 2.01$
1.001	$(1.001, 1.002001)$	$\frac{1.002001-1}{1.001-1} = 2.001$
1.0001	$(1.0001, 1.00020001)$	$\frac{1.00020001-1}{1.0001-1} = 2.0001$
x	(x, x^2)	$\frac{x^2-1}{x-1}$

It seems clear that as x gets closer and closer to 1, the slope of the secant line gets closer and closer to 2. It's worth making sure this would be true for values of x like 0.999 too; but it is. So surely the slope of the tangent line is 2.

Now, what one would really like to do is to get the slope of the tangent line itself by using as the second point $(1, 1)$, so that you go the slope of a line hitting the curve at only 1, point. But if you try this, the slope you compute is $\frac{1-1}{1-1} = \frac{0}{0}$, which is not a number.

This is worth stressing pretty firmly: $\frac{0}{0}$ is not a number. It's not 0, even though 0 over anything except 0 is 0. It's not $\pm\infty$, even though anything but 0 might seem to give $\pm\infty$ when divided by 0. It's not 1, even though anything but 0 divided by itself is 0. It's not a number. In fact, you wouldn't want $\frac{0}{0}$ to be 0 or 1 or $\pm\infty$ in this case—you'd want it to be 2. So it's just as well that—What did we say? $\frac{0}{0}$ is not a number.

Algebraically, then, the story is this. To approximate the slope of the tangent line to $y = x^2$ at the point $(1, 1)$, we take a second point (x, x^2) close to $(1, 1)$ on the curve, and compute the slope of the secant line joining $(1, 1)$ and (x, x^2). This slope is

$$s(x) = \frac{x^2 - 1}{x - 1}.$$

When $x = 1$, this slope is undefined. But when x is close to 1, the slope ought to be close to the slope of the tangent line. So what we want to be able to compute is the value that $s(x)$ gets close to when x is close to 1 but $x \neq 1$. This number is called the *limit* of $s(x)$ as x approaches 1. It is written

$$\lim_{x\to 1} s(x) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1}.$$

In this case, the table above makes a strong case that the value of this limit shold be

$$\lim_{x\to 1} s(x) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = 2.$$

This is what limits are, and this is why we care about them: we need them in order to compute the slopes of curves. We'll see later that we also need limits in order to compute areas of complicated figures, which we'll describe as limits of the areas of simple figures (collections of rectangles) that approximate those complicated figures.

2.2 Pictures of functions with limits

An intuitive way of thinking about limits is to say that $\lim_{x\to a} f(x)$ is the value you expect $f(a)$ to take as long as f doesn't do anything funny at the point $x = a$. There are 3 pictures worth keeping in mind when one thinks about functions having limits. The first, Figure 3, shows a normal, well-behaved function f going smoothly through the point $(2, 1)$. When x is close to 2 but $x \neq 2$, $f(x)$ is close to 1; so $\lim_{x\to 2} f(x) = 1$. Also, $f(2) = 1$.

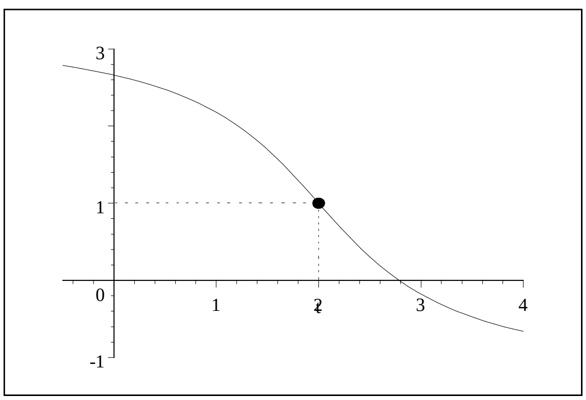

Figure 3: A nice, normal function.

The second figure, Figure 4, shows a function g that is nice and well-behaved everywhere except at $x = 3$. At this one point, the function g is undefined. Traditionally, one represents this with an open circle in the graph. Despite the

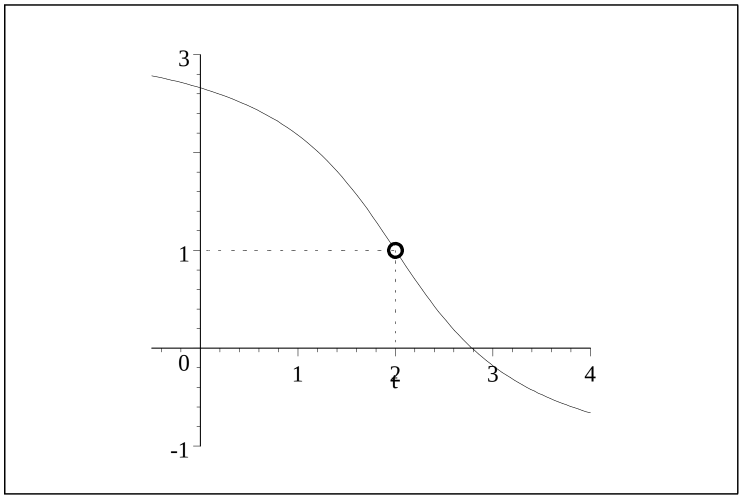

Figure 4: A function undefined at 1 point.

hole, though, it is still the case that when x is close to 2 but $x \neq 2$, $f(x)$ is close to 1; so $\lim_{x\to 2} g(x) = 1$, even though $g(2)$ itself isn't defined.

The last figure, Figure 5 shows an even more pathological function, h. The graph of $y = h(x)$ has a hole at the point $(2, 1)$, but then, unexpectedly, $h(2) = 2$. It is still the case that when x is close to 2 but $x \neq 2$, $h(x)$ is close to 1; so $\lim_{x\to 2} h(x) = 1$, even though $h(2) = 2$.

It would be easy to imagine that the only one of these 3 situations that matters was the one in Figure 3 with the normal function, and that the other two figures are just obnoxious pathologies; but this isn't right. It is true that the situation in Figure 5, where the function h is defined at $x = 2$ but where its value there is unexpected, is an obnoxious pathology—a contrived function of no importance whatsoever. If we only dealt with continuous functions like the one in Figure 3 though, then we wouldn't need the idea of $\lim_{x\to a} f(x)$ at all. We could just use the simpler idea of $f(a)$.

The situation that really matters to us is the one shown in Figure 4, where the function g is undefined at the very point that matters to us. After all, the limit that arose in the slope problem above was

$$\lim_{x\to 1} s(x) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1}.$$

What would this look like graphically? The function $s(x) = \frac{x^2-1}{x-1}$ looks like a straight line, except that it is undefined at the point $(1, 2)$, where the denominator is 0. Figure 6 shows this.

This isn't surprising, since as long as $x \neq 1$, we have

$$s(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1} = x + 1,$$

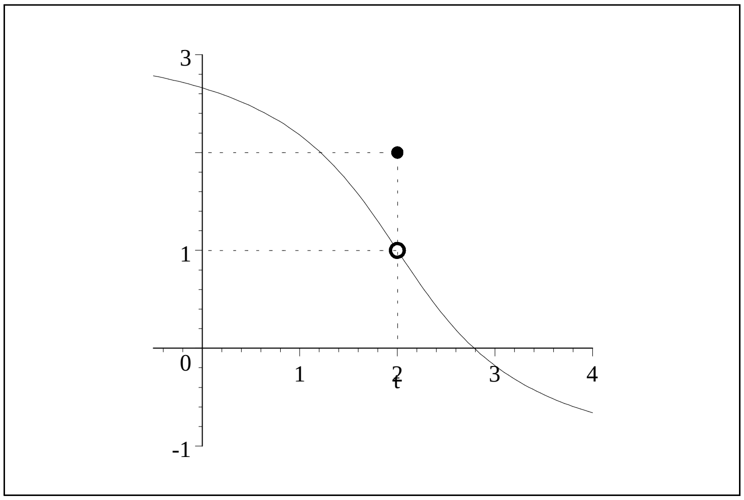

Figure 5: A function with an unexpected value.

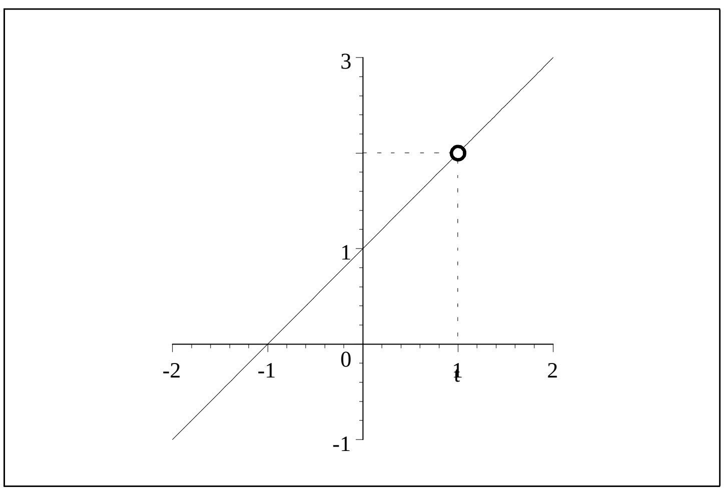

Figure 6: $s(x) = \frac{x^2-1}{x-1}$.

so that the graph of $s(x) = \frac{x^2-1}{x-1}$ should look just like the graph of $y = x + 1$, except that it should be undefined at the point $(1, 1)$. Annoyingly, the one point where $s(x)$ is undefined is the one point where we care about its value.

This is really what limits are for: plugging up single points where functions annoyingly manage to be undefined.

2.3 Pictures of functions without limits

Figures 7, 8, and 9 show 3 ways for a function not to have a limit at a point.

Figure 7 shows a function f for which $\lim_{x\to 1} f(x)$ does not exist. Here the problem is obvious. The function f approaches different limits as $x \to 1$ from the left and as $x \to 1$ from the right; so there is no one limit for the function that applies regardless of direction.

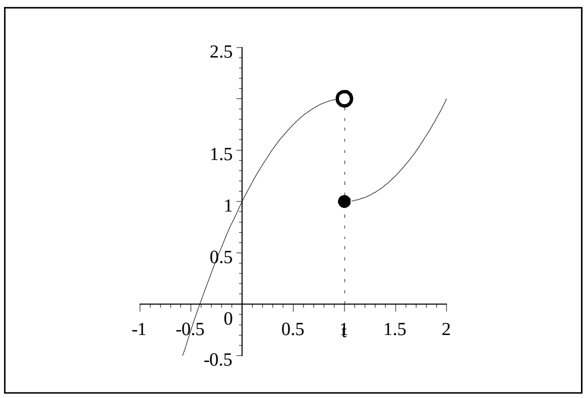

Figure 7: $\lim_{x\to 1} f(x)$ does not exist.

Figure 8 also shows a function which has no limit as $x \to 1$, and here again, the problem is obvious: the function has a vertical asymptote at the point $x = 1$; so it cannot have a limit at this point.

Figure 9, which shows a function $h(x)$ with no limit as x approaches 0, is more subtle. The function here is $h(x) = \sin(1/x)$. This function has infinitely many peaks and valleys which get closer and closer together as x approaches 0 from either side. There is therefore not any one number that $h(x)$ approaches as $x \to 0$. Maple writes the limit in this case as $-1..1$, meaning that every number between -1 and 1 is a limit point of the function. This is non-standard, but informative. Standard mathematical usage would just be to say that the limit does not exist.

2.4 A formal definition of limits

The informal definition of $\lim_{x\to a} f(x) = L$ is that when x is close to a but $x \neq a$, then $f(x)$ is close to L. More precisely, we can say that we can make

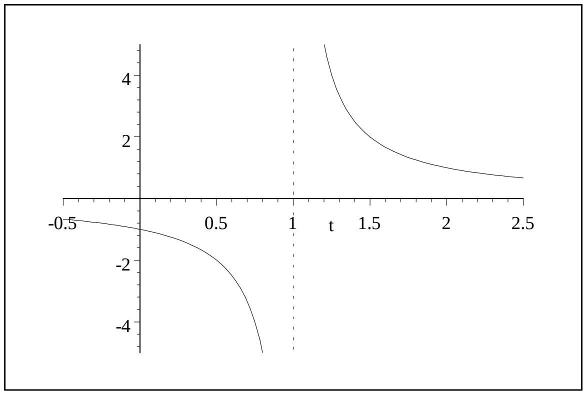

Figure 8: $\lim_{x\to 1} g(x)$ does not exist.

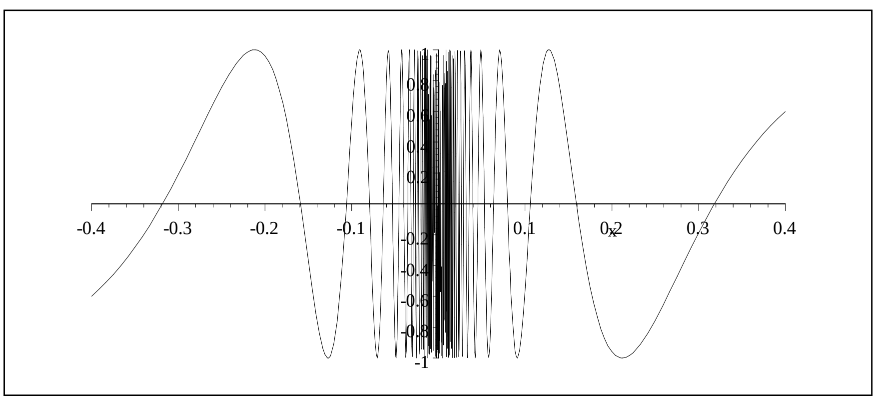

Figure 9: $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

$f(x)$ as close to L as we like, as long as we make x close enough to a. This is the content of the official definition of $\lim_{x\to a} f(x) = L$, which says that it means that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$. In other words, no matter how close, ε, you want to make $f(x)$ to L, you can always get it that close as long as you pick x close enough to a, at most a distance δ away. The geometric idea is shown in Figure 10. No matter how small the distance ε is chosen to be, one has to be able to find a distance δ such that every point within a distance δ of a (except for maybe a itself) is mapped by f into the interval $L - \varepsilon < y < L + \varepsilon$.

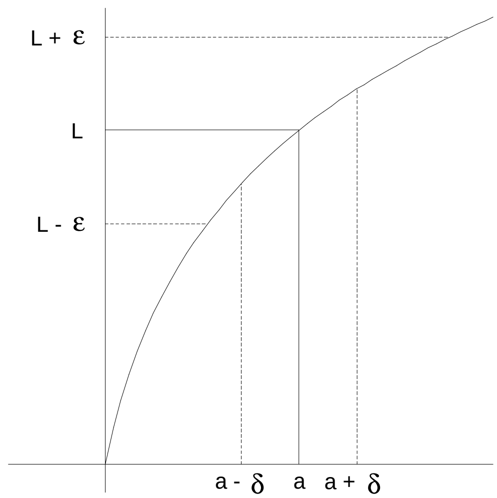

Figure 10: Limits in terms of ε and δ.

2.5 Simple algebraic computations of limits.

We'll need to spend some time working out methods for computing limits, but there are a few things we can say right away.

There are simple rules for computing the limits we don't care about, which are often belabored by calculus books. These rules basically say that the value of the limit of a simple function without holes in its graph is the value of the function at that point.

If c is a constant, then $\lim_{x\to a} c = c$. (No surprise, right? When x gets close to a, c gets close to c.)

$\lim_{x\to a} x = a$. (And when x gets close to a, x gets close to, let's see... a.)

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ exist, then

$$\lim_{x\to a} (f(x) + g(x)) = \left(\lim_{x\to a} f(x)\right) + \left(\lim_{x\to a} g(x)\right) = L + M.$$

The same rule works for $f(x) - g(x)$ and for $f(x)g(x)$. It also works for $f(x)/g(x)$ as long as $M \neq 0$.

Armed with these rules, we can do limits like

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2+5}{x-1} &= \frac{\lim_{x\to 2}(x^2+5)}{\lim_{x\to 2}(x-1)} \\
&= \frac{\lim_{x\to 2}(x^2) + \lim_{x\to 2} 5}{\lim_{x\to 2} x - \lim_{x\to 2} 1} \\
&= \frac{(\lim_{x\to 2} x)\,(\lim_{x\to 2} x) + \lim_{x\to 2} 5}{\lim_{x\to 2} x - \lim_{x\to 2} 1} \\
&= \frac{2\cdot 2+5}{2-1} \\
&= \frac{2^2+5}{2-1} = 9.
\end{aligned}$$

These limits don't interest us very much.

The more important observation is that changing the value of a function at one point doesn't change the value of any limit involving that function. If you are computing the limit as $x \to a$, then changing the value of $f(a)$ it makes no difference, because the limit depends only on what happens when x is close to a but $x \neq a$. Changing the value of $f(b)$ for any $b \neq a$ doesn't change the limit, since the limit depends of the value of $f(x)$ for x very close to a, closer than any fixed b. We can therefore do things like this:

The functions $s(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}$ and $t(x) = x+1$ differ only at one point: $s(1)$ is undefined, while $t(1) = 2$. It must therefore be the case that $s(x)$ and $t(x)$ have the same limit at every point; so

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = \lim_{x\to 1} x + \lim_{x\to 1} 1 = 1+1 = 2.$$

At this stage, we have finally computed the slope of the tangent line to $y = x^2$ at the point $x = 1$ in complete detail, and we can be certain of the value, 2, that we conjectured at the beginning of our discussion of limits.

Now might be a good moment to reread the discussion of slopes at the beginning of this section, and to make sure it makes sense.

2.6 Algebraic computation of more complicated limits

2.6.1 Review.

At this stage, we have 2 principles for computing $\lim_{x\to a} f(x)$.

(1) Compute $f(a)$. If you get a meaningful value, and if is f is nice and continuous, then $\lim_{x\to a} f(x) = f(a)$.

(2) If when you try to compute $f(a)$, you get $\frac{0}{0}$, then look for and remove common factors in the numerator and denominator of $f(x)$, and try again.

An example of the first process would be

$$\lim_{x\to 3}\frac{x^2-2x+5}{3x-1}=\frac{3^2-2(3)+5}{3(3)-1}=\frac{8}{8}=1.$$

The details here could be justified by our theorems about the limit of a sum, difference, product, and quotient, and about the limit of a constant and the limit of x.

An example of the second process would be

$$\lim_{x\to 2}\frac{x^3-8}{2x-4}.$$

If we try just plugging in $x=2$, we get $\frac{2^2-8}{2(2)-4}=\frac{0}{0}$, which is undefined. We therefore try to find and remove common factors from the numerator and denominator. Since the denominator is $2(x-2)$, we even know what common factor we want to find and eliminate: $x-2$. We could therefore either remember how the numerator factors, or we could divide the numerator by $x-2$ and see what emerges. We would find that $x^3-8=(x-2)(x^2+2x+4)$, so that the limit becomes

$$\lim_{x\to 2}\frac{x^3-8}{2x-4}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{2(x-2)}=\lim_{x\to 2}\frac{x^2+2x+4}{2}=\frac{2^2+2(2)+4}{2}=6.$$

It's worth thinking a moment about what has happened here conceptually. When we cancel the common factor of $x-2$ to move from $\frac{(x-2)(x^2+2x+4)}{2(x-2)}$ to $\frac{x^2+2x+4}{2}$, we don't quite get the same function back, since $\frac{x^2+2x+4}{2}$ is defined for every real number x, while $\frac{(x-2)(x^2+2x+4)}{2(x-2)}$ is undefined at $x=2$. Except at this one one point, however, the two functions agree everywhere. We already saw, though, that changing the value of a function at one point does not change any limit involving that function; so that the calculation above makes sense.

2.6.2 Another example of eliminating common factors.

Sometimes you just need to do algebra to remove common factors in the numerator and denominator. Consider, for instance,

$$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}.$$

Almost always, I would start such a problem by rewriting the fractions to be on 2 levels rather than 3. It's too hard to think about otherwise. You get

$$\lim_{x\to 3}\left(\frac{1}{x-3}\left[\frac{1}{x}-\frac{1}{3}\right]\right).$$

The only thing that comes to mind is to put everything over a common denominator and to hope for the best. You get

$$\begin{aligned}\lim_{x\to 3}\left(\frac{1}{x-3}\left[\frac{3}{3x}-\frac{x}{3x}\right]\right) &= \lim_{x\to 3}\left(\frac{1}{x-3}\left[\frac{3-x}{3x}\right]\right)\\ &= \lim_{x\to 3}\left(-\frac{1}{3x}\right) = -\frac{1}{9}.\end{aligned}$$

2.6.3 The trick with square roots.

Consider

$$\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}.$$

If we start out and plug in $x = 3$, we get $\frac{0}{0}$. But how can we remove a common factor from the numerator and denominator? One standard answer is to multiply by 1 in a complicated way:

$$\begin{aligned}\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3} &= \lim_{x\to 3}\left(\frac{\sqrt{x}-\sqrt{3}}{x-3}\cdot\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}\right)\\ &= \lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})}\\ &= \lim_{x\to 3}\frac{1}{(\sqrt{x}+\sqrt{3})} = \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.\end{aligned}$$

2.6.4 Another view of the trick with square roots.

Some people find it simpler to think of the example above like this: The denominator, $x-3$, is actually the difference of two squares,

$$x-3 = (\sqrt{x})^2-(\sqrt{3})^2.$$

It can therefore be factored as $x-3 = (\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})$. Our limit is therefore

$$\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3} = \lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})} = \lim_{x\to 3}\frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

Use whichever of these tricks seems most natural to you.

2.6.5 The Squeeze Theorem.

If there are 3 functions, f, g, h defined near a, and if at every point except maybe at a we have $f(x) < g(x) < h(x)$, and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} g(x) = L$ as well.

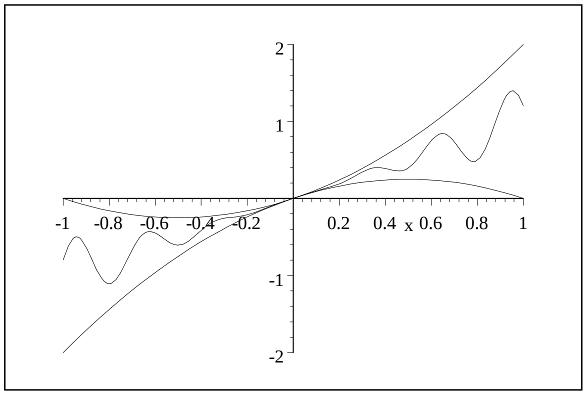

Figure 11: The Squeeze Theorem.

In the normal applications of this theorem, f and h are simple functions whose limits we can compute, and g is a complicated function we can't analyze directly. Figure 11 shows the idea.

To see that the Squeeze Theorem is sometimes necessary, consider the function $f(x) = x\sin(\frac{1}{x})$, shown in Figure 12.

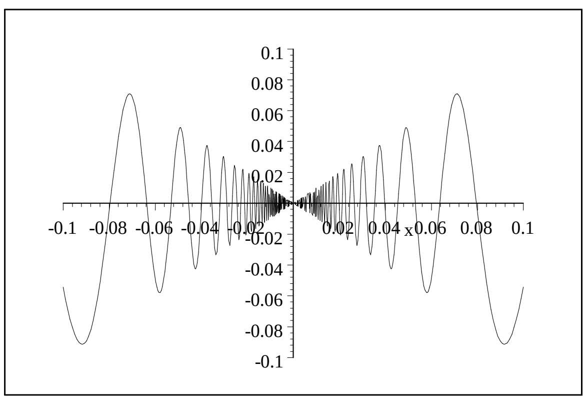

Figure 12: $f(x) = x\sin(\frac{1}{x})$.

We would like to be able to say that since $\lim_{x\to 0} x = 0$, it follows that

$$\lim_{x\to 0}\left(x\sin\left(\frac{1}{x}\right)\right) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\sin\left(\frac{1}{x}\right)\right) = 0\cdot\left(\lim_{x\to 0}\sin\left(\frac{1}{x}\right)\right) = 0.$$

This isn't valid, however, since we've already seen that $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ doesn't exist. While any number times 0 is guaranteed to be 0, we don't know anything about what the product of 0 and some non-existent thing might be. For instance, it would clearly be invalid to say

$$\lim_{x\to 0} x\frac{1}{x} = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \frac{1}{x}\right) = 0 \cdot \left(\lim_{x\to 0} \frac{1}{x}\right) = 0,$$

since $\lim_{x\to 0} x\frac{1}{x} = \lim_{x\to 0} 1 = 1$.

The ticket to computing $\lim_{x\to 0}\left(x \sin\left(\frac{1}{x}\right)\right)$ is the Squeeze Theorem. We know that as long as $x \neq 0$,

$$\begin{aligned} -1 &\leq \sin(1/x) \leq 1 \\ -|x| &< x\sin(1/x) < |x|. \end{aligned}$$

It is also clear that

$$\lim_{x\to 0}(-|x|) = 0 = \lim_{x\to 0}|x|,$$

so by the Squeeze Theorem,

$$\lim_{x\to 0}\left(x\sin\left(\frac{1}{x}\right)\right) = 0.$$

2.6.6 Two tricky trig limits.

Plotting points or making a table of values makes it appear that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

and that

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}.$$

Later, we'll need these limits for a couple of technical calculations. Now, let's see why they are true. If you plug in $x = 0$ to the first function, you get $\frac{\sin 0}{0} = \frac{0}{0}$, an undefined result. Our general approach would say to remove a common factor between the numerator and denominator. But how can we factor x out from $\sin x$? There is no obvious way.

The trick turns out to be a clever use of the Squeeze Theorem, along with some smart geometry. First of all, think about out standard drawing of a point on the unit circle, in Figure 13.

If θ is the angle shown, then the length of the arc marked out by this angle is also θ. The length of the vertical side of the triangle shown is $\sin\theta$. It is clear geometrically that the arc is longer than the side of the triangle; so $\sin\theta < \theta$, or $\frac{\sin\theta}{\theta} < 1$.

Now look at a less obvious bit of geometry. Figure 14 shows the same angle θ along with a triangle formed by a vertical line through $(1, 0)$.

The area of the whole unit circle of 2π radians is π, and the area of the "pie slice" marked by angle θ must be the area of the whole circle, π, multipled by the fraction of the whole circle that lies inside the pie slice, $\frac{\theta}{2\pi}$. In other words, the area of the pie slice is $\pi\left(\frac{\theta}{2\pi}\right) = \frac{\theta}{2}$. Since $\tan\theta = \frac{opp}{adj}$, the vertical side of the

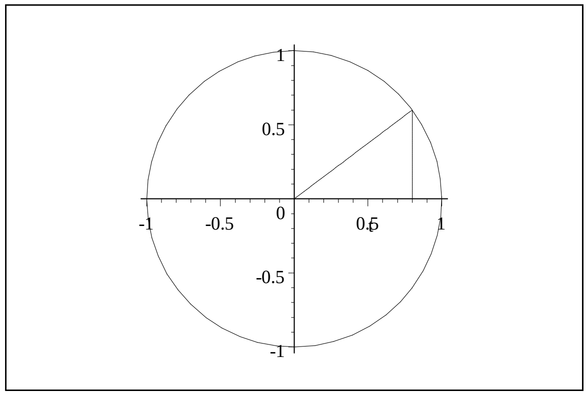

Figure 13: A triangle in the unit circle.

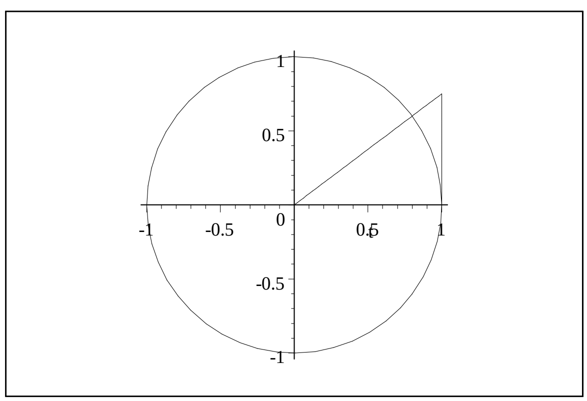

Figure 14: A triangle outside the unit circle.

triangle must be $\tan\theta$. (Incidentally, this is where the tangent function got its name: it's the length of the vertical segment that is tangent to the circle.) The area of the triangle is therefore

$$\frac{1}{2}bh = \frac{1}{2}\cdot 1\cdot \tan\theta = \frac{\tan\theta}{2}.$$

It is clear geometrically that the triangle is bigger than the "pie slice," so it must be that

$$\begin{aligned}
\frac{\theta}{2} &< \frac{\tan\theta}{2} \\
\theta &< \tan\theta = \frac{\sin\theta}{\cos\theta} \\
\frac{\sin\theta}{\theta} &> \cos\theta.
\end{aligned}$$

Putting this together with the previous inequality gives

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Since $\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0} 1 = 1$, it follows from the Squeeze Theorem that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

This was the first of the two clever trig limits. The second can be gotten algebraically from the first. We use a trick like the square root trick to observe that

$$\frac{1-\cos\theta}{\theta^2} = \frac{1-\cos\theta}{\theta^2}\cdot\frac{1+\cos\theta}{1+\cos\theta} = \frac{1-\cos^2\theta}{\theta^2(1+\cos\theta)} = \frac{\sin^2\theta}{\theta^2(1+\cos\theta)}.$$

This means that

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} &= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta^2(1+\cos\theta)} \\
&= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{1}{1+\cos\theta}\right) \\
&= 1\cdot 1\cdot\frac{1}{2} = \frac{1}{2},
\end{aligned}$$

which is the second clever trig limit. It is worth stating a corollary, which is the result we actually need,

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta^2}\cdot\theta\right) \\
&= \left(\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}\right)\left(\lim_{\theta\to 0}\theta\right) \\
&= \frac{1}{2}\cdot 0 = 0.
\end{aligned}$$

2.7 Continuity; One-sided and Infinite Limits

2.7.1 Continuity at a point.

Intuitively, a function is continuous if you can draw the graph without picking up your pencil. The idea of limits lets us take this vague idea and render it precise. Here's the official definition:

The function f is continuous at the point $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$. This means 3 things:

(1) $\lim_{x\to a} f(x)$ has to exist.

(2) $f(a)$ has to exist.

(3) $\lim_{x\to a} f(x)$ and $f(a)$ have to be equal.

There are therefore 3 ways a function could go about not being continuous at the point $x = a$: any of the 3 conditions in the definition could be false.

Let's try to convince ourselves that the definition here captures the idea of continuity. First of all, it's easy to see that normal functions at normal points are continuous. For instance, if $f(x) = x^2$, then f is continuous at $x = 2$ (or at any other point, for that matter) because

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 = 4 = 2^2 = f(2).$$

In general, if you can compute $\lim_{x\to a} f(x)$ by just plugging $x = a$ into $f(x)$, then f is continuous at a.

2.7.2 Discontinuous functions.

Since so many normal functions are continuous everywhere, we may gain more insight into what continuity means by studying the discontinuous functions. That is, we'll try to learn about what continuity is by learning about what continuity isn't. A function is discontinuous at $x = a$ if it fails to satisfy one or more of the conditions above. We have to look a bit to find such functions, but let's locate a bunch of these discontinuous functions and see that they also fail to satisfy our intuitive ideas about continuity.

Functions discontinuous because $\lim_{x\to a} f(x)$ does not exist. The functions pictured in Figures 7, 8, 9 fail to be continuous at the points at which they do not have limits. That is, a function with a break at $x = a$ where it approaches different limits from the left and the right fails to be continuous at $x = a$. Similarly, a function with a vertical asymptote at $x = a$ fails to be continuous at $x = a$, and a function with infinitely many waves of the same height in every neighborhood of $x = a$ fails to be continuous at a.

Functions discontinuous because $f(a)$ does not exist. The function pictured in Figure 4, which is a smooth function except that it is undefined at one point, also fails to be continuous at the point $x = 2$ where it is undefined. Although $\lim_{x\to 3} f(x) = 1$, $f(2)$ is undefined, and cannot equal the limit.

Functions where $\lim_{x\to a} f(x) \neq f(a)$. Finally, for completeness, the function in Figure 5, which approaches 1 as $x \to 2$, but which then has a jump at this one point so that there is a hole in the graph at $(2, 1)$ and a single dot at $(2, 2)$, is discontinuous at $x = 2$, since even though both the limit and the function are defined at $x = 2$, we have

$$\lim_{x\to 2} f(x) = 1 \neq 2 = f(2).$$

All of these functions seem like they should be discontinuous at these points using our intuitive notion of discontinuous functions as ones where you have to pick up your pencil to draw the graph; so the formal definition of continuous functions seems like a good one.

2.7.3 Continuity in an interval.

The definition of a continuous function talked about continuity at one point. What we are more often interested in is continuity in an interval. A function f is continuous on an interval if and only if it is continuous at every point in the interval. It is continuous, period, if it is continuous everywhere.

2.7.4 Easy Theorems on Continuity.

The theorems about the sum, product, difference, and quotient of limits being the limits of the sum, product, difference, and quotients of the functions involved give rise to easy theorems saying that the sum, product, difference, and quotient of 2 continuous functions are themselves continuous. For instance, if f and g are continuous at a, then

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = f(a)g(a),$$

so fg is continuous at a. By the same reasoning, $f + g$ and $f - g$ are continuous at a, as is f/g if $g(a) \neq 0$.

2.7.5 Intermediate Value Theorem.

If f is continuous on a closed interval $[a, b]$, and if k is any number between $f(a)$ and $f(b)$, then there is some point c, $a < c < b$, for which $f(c) = k$.

For instance, Figure 15 shows a function $f(x) = (x - 2)^3 - 2(x - 2) + 4$. This function is a polynomial, so it is continuous at every point on $[0, 4]$. At the endpoints, $f(0) = 0$ and $f(4) = 8$. If we let $k = 4$, then $f(0) = 0 < k = 4 < f(4) = 8$; so by the IVT, some point c can be found at which $f(c) = 4$. In fact, it appears that there are 3 such points.

On an intuitive level, the IVT is an obvious fact. If you start out on one side of $y = k$ and you draw at curve without picking up your pencil, and you end up on the other side of $y = k$, then at some point in between, you must have crossed $y = k$. It turns out, though, that it is not at all trivial to give a formal proof of this fact from the usual axioms for the real numbers, and using

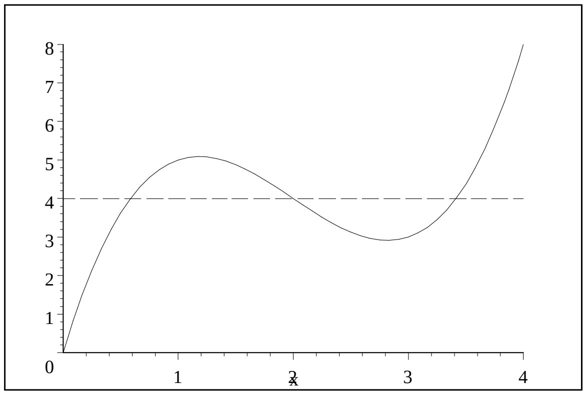

Figure 15: $f(x) = (x-2)^3 - 2(x-2) + 4$.

the formal definition of continuity instead of our informal notion. In the book I learned calculus from, the IVT was in a chapter refreshingly entitled, "Three Hard Theorems."

2.8 Extensions of the Limit Concept

2.8.1 One-sided limits.

Let $f(x)$ be the function shown above in Figure 7. Then $\lim_{x\to 1} f(x)$ does not exist, since there is single number that the values of $f(x)$ are approaching as x approaches 1. On the other hand, if x is close to 1 and $x < 1$, then the graph shows that $f(x)$ is close to 2. We describe this situation, that $f(x)$ is close to 2 whenever x is close to 1 and $x < 1$, by writing

$$\lim_{x\to 1^-} f(x) = 2.$$

Similarly, the graph shows that $f(x)$ is close to 1 whenever x is close to 1 and $x > 1$. We write this as

$$\lim_{x\to 1^+} f(x) = 1.$$

These two quantities, $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$, are called the limit of $f(x)$ as x approaches a from the left (or from below) and the limit of $f(x)$ as x approaches a from the right (or from above). There's no very subtle idea here—it's just notation. Obviously if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L,$$

then $f(x)$ gets close to L whenever x gets close to a, whether from the right or from the left; so that

$$\lim_{x\to a} f(x) = L$$

as well. Equally obviously, if

$$\lim_{x\to a} f(x) = L,$$

then

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

It's worth being explicit that the limit as x approaches a from below means the limit as x approaches a from smaller values of x, not of y. In the function in Figure 7, $\lim_{x\to 1^-} f(x)$ is 2, not 1.

It's also worth noting that more obnoxious functions like the one in Figure 9 above may lack one-sided limits: none of the limits

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0^+} \sin\left(\frac{1}{x}\right), \qquad \lim_{x\to 0^-} \sin\left(\frac{1}{x}\right)$$

exist.

2.8.2 Vertical Asymptotes and Infinite Limits.

We've so far looked a lot at computing limits of functions that look like

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{a(x)}{b(x)}.$$

Our basic rules have been to try to just plug in $x = c$ and get $a(c)/b(c)$. If this is a number, and if the numerator and denominator are continuous functions (polynomials, for example), then

$$\lim_{x\to c} \frac{a(x)}{b(x)} = \frac{a(c)}{b(c)},$$

and we're done.

If, on the other hand, we plug in $x = c$ and we get $\frac{a(c)}{b(c)} = \frac{0}{0}$, which is not a number, then we have a whole repertoire of tricks to find and eliminate common factors between the numerator and denominator, after which we plug in $x = c$ again, and see what happens.

But what if we plug in $x = c$ and we get not $\frac{a(c)}{b(c)} = \frac{2}{3}$ or $\frac{a(c)}{b(c)} = \frac{0}{0}$, but $\frac{a(c)}{b(c)} = \frac{3}{0}$, or, in general, $\frac{a(c)}{b(c)} = \frac{p}{0}$, where p is a nonzero number? Assume both the numerator and denominator are continuous at c. Then when x is a number close to c, but $x \neq c$, $a(x)$ will be close to p and $b(x)$ will be close to 0. The result will be that $\frac{a(x)}{b(x)}$ will have a normal sized numerator and a tiny denominator, so that $\frac{a(x)}{b(x)}$ will either be huge and positive (like 10^{100}), or huge and negative (like -10^{100}). The closer you take x to c, the closer the denominator will get to 0, and the bigger the fraction will get in absolute value. In other words, $\frac{a(x)}{b(x)}$ will have a vertical asymptote at $x = c$.

There are two possible things one could therefore say about $\lim_{x\to c}\frac{a(x)}{b(x)}$. One perfectly honest analysis would simply be to say that the limit does not exist. This is absolutely correct based on the proper definition of limits in terms of ε and δ. But it's really a lot more informative to try to determine whether $\frac{a(x)}{b(x)}$ is big and positive or big and negative on each side of $x = c$. Are we, for instance, in a situation like $f(x) = \frac{1}{x}$ in Figure 16, where, with the obvious meaning for the symbols,

$$\lim_{x\to 0^+}\frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x\to 0^-}\frac{1}{x} = -\infty,$$

or are we in a situation like $f(x) = \frac{1}{x^2}$ in Figure 17, where

$$\lim_{x\to 0^+}\frac{1}{x^2} = +\infty \quad \text{and} \quad \lim_{x\to 0^-}\frac{1}{x^2} = +\infty.$$

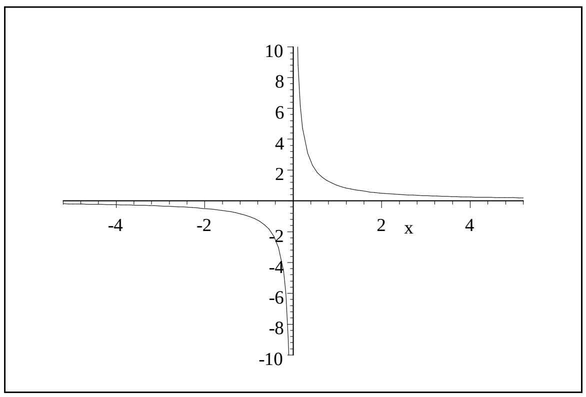

Figure 16: $f(x) = \frac{1}{x}$.

Normally, the way to figure out which is the case is to plug in numbers close to c, either numerically or conceptually. That is, you could guess that

$$\lim_{x\to 0^-}\frac{1}{x} = -\infty$$

because $\frac{1}{0.0001} = -10000$, or you could say that conceptually, in computing the limit you are doing something like

$$\frac{1}{-tiny} = Huge.$$

To pick a more complex example, suppose we wanted to compute

$$\lim_{x\to 2^+}\frac{x^3 - 7x + 11}{4x^2 - 20x + 24}.$$

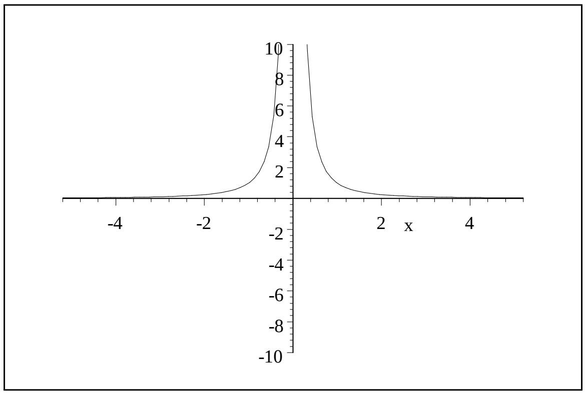

Figure 17: $f(x) = \frac{1}{x^2}$.

We would start by plugging in $x = 2$, which would make the fraction into $\frac{5}{0}$. This would tell us that we have a vertical asymptote at $x = 2$. Knowing that the denominator is 0 when $x = 2$, we would naturally factor $(x-2)$ out of the denominator to get

$$\lim_{x\to 2^+} \frac{x^3 - 7x + 11}{4(x-2)(x-3)}.$$

If x is slightly larger than 2, then the fraction is roughly

$$\frac{5}{4(+tiny)(-1)} = -\frac{5}{4} \cdot \frac{1}{+tiny} = -\frac{5}{4}(+Huge) = -Huge,$$

so

$$\lim_{x\to 2^+} \frac{x^3 - 7x + 11}{4x^2 - 20x + 24} = -\infty.$$

By the same reasoning, if x is slightly smaller than 2, then the fraction is roughly

$$\frac{5}{4(-tiny)(-1)} = -\frac{5}{4} \cdot \frac{1}{-tiny} = -\frac{5}{4}(-Huge) = +Huge,$$

so

$$\lim_{x\to 2^+} \frac{x^3 - 7x + 11}{4x^2 - 20x + 24} = +\infty.$$

2.8.3 Combining techniques.

One final example:some of the cases we have looked at can appear in the same problem. For instance, suppose we wanted to compute

$$\lim_{x\to 1} \frac{x^2 + 2x - 3}{x^2 - 2x + 1}.$$

We would start out by optimistically plugging in $x = 1$, and would find that we got $\frac{0}{0}$ as a result. We therefore look for common factors, knowing that $x - 1$ is the prime suspect. Factoring out an $x-1$ from the numerator and denominator turns the fraction into

$$\frac{x^2+2x-3}{x^2-2x+1} = \frac{(x-1)(x+3)}{(x-1)^2} = \frac{x+3}{x-1}.$$

We therefore know that

$$\lim_{x\to 1} \frac{x^2+2x-3}{x^2-2x+1} = \lim_{x\to 1} \frac{x+3}{x-1},$$

and we set about computing this second limit.

Again we plug in $x=1$, and this time we get $\frac{5}{0}$, which tells us the function has a vertical asymptote at $x=1$. So now we imagine plugging in a number x just slightly bigger than 1. We would get

$$\frac{x+3}{x-1} \doteq \frac{4}{+tiny} = +Huge,$$

so that

$$\lim_{x\to 1^+} \frac{x^2+2x-3}{x^2-2x+1} = \lim_{x\to 1^+} \frac{x+3}{x-1} = +\infty.$$

Similarly, if we imagine plugging in a number x just slightly smaller than 1, we find

$$\frac{x+3}{x-1} \doteq \frac{4}{-tiny} = -Huge,$$

so that

$$\lim_{x\to 1^-} \frac{x^2+2x-3}{x^2-2x+1} = \lim_{x\to 1^-} \frac{x+3}{x-1} = -\infty.$$