

4 Derivatives and Graphing

One of the traditional big uses for derivatives is as a tool in understanding and graphing functions. It would be easy to imagine that in this day of computer algebra systems and graphing calculators, this tool of the 17th Century had become obsolete. I think this is false. Let me try to illustrate just how completely one can be misled by mechanical graphing tools. My particular example has been run on Maple, but it's easy enough to find curves that defeat the mechanical graphing tool of your choice.

We'll start with a function that is slightly messy, but really not all that complicated.

$$f(x) = e^{-x^2/25,000,000} \left[\sin\left(\frac{x}{50}\right) - \frac{1}{(2000x - 250,000)^2} \right].$$

Here $e \doteq 2.718281828459$ is a special number, the base of the natural logs, that we will meet more carefully down the road a bit. For right now, all that matters is that it's some number between 2 and 3.

The question is, what does the graph of $y = f(x)$ look like? In particular, what are the maximum and minimum values attained by the function?

The attached Maple Worksheet looks at this function in a bunch of different plot windows, which I hope should show you just how much one's view of a function depends on the window you choose, how hard it is to pick the right window. One also often needs more than one window in order to illustrate all the important properties of a function. For the record, the windows chosen in the Maple worksheet were

Figure	Plot window	Apparent shape
1	$-20 < x < 20$	Diagonal, $y = -\infty$ to $+\infty$
2	$-90 < x < -70$	Parabola, $y = -1$ to $+\infty$
3	$-90 < x < -70, -10 < y < 10$	Horiz. line near $y = -1$
4	$-100 < x < 100$	Curve
5	$-1000 < x < 1000$	Sine wave, $y = -1$ to 1
6	$-10,000 < x < 10,000$	Wave packet
7	$-20,000 < x < 20,000$	Wave packet
8	$100 < x < 200$	Diagonal line
9	$120 < x < 130$	Line with a blip
10	$124 < x < 126$	Line with a long blip
11	$124.9 < x < 125.1$	Vert. asymp. to $y = -\infty$
12	$124.99 < x < 125.01, -5 < y < 1$	Vert. asymp. to $y = -\infty$

What this example ultimately shows is the importance of being able to look at a function analytically in order to determine where its interesting features are. Even in the world of Maple and graphing calculators, calculus still has a role to play in graphing.

Incidentally, you might be interested to know how I came up with this weird function. You know what the graph of $y = \sin x$ looks like. Dividing x by 50 to get $y = \sin(x/50)$ stretches the graph by a factor of 50 in the horizontal

direction, yielding very slow waves. The function $y = e^{-x^2}$ is a bell curve, related to the bell curve used in statistics, and shown below in Figure 1.

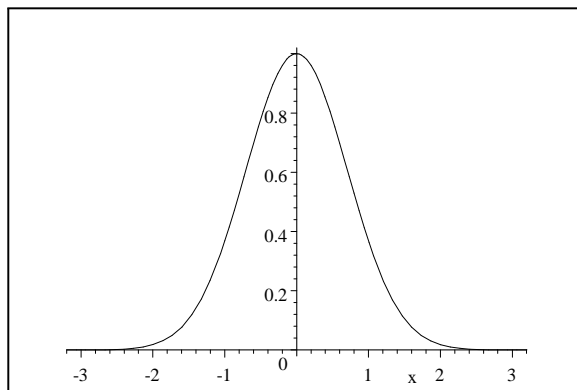


Figure 1: $y = e^{-x^2}$.

Dividing x by 5000 to get $e^{-x^2/25,000,000}$ stretches this curve by a factor of 5000 in the horizontal direction. When we multiply this by the sine curve $y = \sin(x/50)$, we get a wave oscillating between the bottom and top of a wide envelope made by this stretched bell curve and its negation. Finally, the function $y = -\frac{1}{(x-125)^2}$ has a vertical asymptote at $x = 125$. By replacing this with $-\frac{1}{(2000x-250,000)^2}$, we get a function that still has a vertical asymptote at this point, but an exceedingly narrow one that Maple misses unless we zoom way in.

4.1 Derivatives, slopes, and extrema: a preview

We already know a good bit about the connection between derivatives and graphing. A positive derivative and an increasing function go together, as do a negative derivative and a decreasing function. A derivative of zero means a horizontal tangent line, which happens at maxima and minima of f . There are some details that aren't quite right in this rough summary, though. For instance, the function $f(x) = |x|$ has a minimum at $x = 0$, which is not a point where $f'(x) = 0$, but a point where $f'(x)$ does not exist. Similarly, $g(x) = x^3$ has $g'(0) = 0$, but it doesn't have a maximum or a minimum at $(0, 0)$. It has a horizontal tangent line at $(0, 0)$, but it is increasing everywhere in the sense that whenever $a < b$, it follows that $g(a) < g(b)$. A good bit of our work for the next little while will be devoted to getting all the details right in this association between, on one hand, points where the derivative is positive, negative, or zero, and, on the other hand, points where the function is increasing, decreasing, or at an extremum.

4.2 Global and local extrema

If we are plotting a function $y = f(x)$ on an interval I , one thing that naturally interests us is where the graph reaches its maximum and minimum values. A point at which f reaches the lowest value it attains anywhere on I is called a minimum point of f on I , or, to be more precise, a global or absolute minimum. Similarly, a global maximum is a point where f attains the largest value it attains anywhere on the interval I . A function can have more than one global maximum or minimum: think about $f(x) = \sin x$ on the interval $0 \leq x \leq 2\pi$. Maxima and minima are collectively called extrema.

A local or relative minimum is a point other than an endpoint of I at which the graph is lower than it is at any other nearby point. It need not be a global minimum, however. Local maxima are defined the same way as local minima. On the surface of the earth, the elevation function has a global maximum at the top of Mount Everest; the top of every mountain or hill is a local maximum. As an example, consider the function $y = x^4 - 8x^2$ on the interval $I = [-2, 3]$, which is shown in Figure 2.

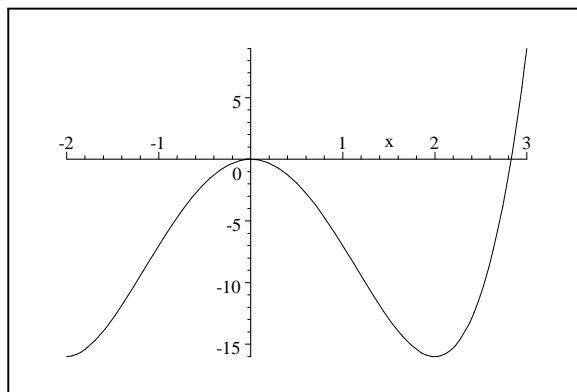


Figure 2: $y = x^4 - 8x^2$

This function has a local minimum at $(2, -16)$ and a local maximum at $(0, 0)$. The global maximum is at $(3, 9)$; there are global minima at $(-2, -16)$ and at $(2, -16)$.

Clearly the only points that could be global extrema of f on an interval I are local extrema of f and the endpoints of the interval I ; but what can we say about the local extrema of f ?

Let x_0 be a local maximum of f . Then there exists an interval $[x_0 - \varepsilon, x_0 + \varepsilon]$ in which every point x satisfies $f(x) \leq f(x_0)$. Let $0 < h < \varepsilon$. Then

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

Since this holds for every small, positive h , it follows that

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0,$$

if the limit exists. Similarly, if $-\varepsilon < h < 0$, then

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

Since this holds for every small, negative h , it follows that

$$\lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \leq 0,$$

if it exists. Since the limit from one side is negative and the limit from the other side is positive, it follows that if

$$\begin{aligned} f'(x_0) &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \end{aligned}$$

exists, then $f'(x_0) = 0$.

We have therefore shown that every local minimum occurs either at a point x_0 where $f'(x_0) = 0$, or at a point x_0 where $f'(x_0)$ is undefined. The same thing holds of local minima, either by using the same proof, or by using the fact that a local minimum of $y = f(x)$ is a local maximum of $y = -f(x)$.

A point at which $f'(x)$ is either zero or undefined is called a critical point of f . What we have therefore shown is that every local extremum of f on the interval I is a critical point of f , and that every global extremum is either a critical point of f or an endpoint of I .

4.3 Examples

4.3.1 Example 1

Suppose we are asked to find all local and global extrema of the function $f(x) = 3x^5 - 5x^3$ on the interval $I = [-2, 2]$. We know that

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x - 1)(x + 1).$$

The derivative is defined at every point x in the interval I ; so the only critical points are the points at $x = 0$, $x = 1$, and $x = -1$, where $f'(x) = 0$.

Is there a way to tell whether each of these points is a local maximum or minimum or neither without having the graph of the function? Yes. By looking

at where each of the factors 15 , x^2 , $x + 1$, and $x - 1$ of $f'(x)$ is positive, negative, or zero, we can see that

$$\begin{aligned} f'(x) &> 0 \text{ if } x < -1 \text{ or } x > 1 \\ f'(x) &\leq 0 \text{ if } -1 \leq x \leq 1. \end{aligned}$$

This means that the point at $x = -1$ must be a local maximum, that the point at $x = 1$ must be a local minimum, and that the point at $x = 0$ must be neither.

To get the global extrema, recall that a global extremum must be either a local extremum or an endpoint of the interval in question. The 4 possible global extrema are therefore the 4 points listed in the following table.

x	$f(x)$	Nature
-2	-56	Global min.
-1	2	
1	-2	
2	56	Global max.

Do you now have a clear picture in your mind of the graph of $y = f(x)$? If not, then try to pull together everything we've just learned and to develop such an image. The actual plot is in Figure 3.

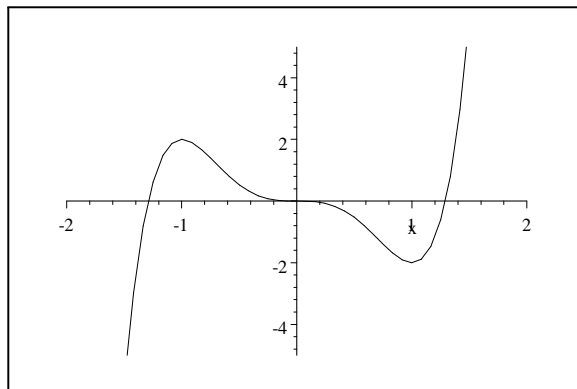


Figure 3: $y = 3x^5 - 5x^3$

4.3.2 Example 2

Suppose we are asked to find all local and global extrema of the function $f(x) = x - 6x^{2/3}$ on the interval $I = [-4, 125]$. We know that

$$f'(x) = 1 - 4x^{-1/3},$$

which is zero when

$$\begin{aligned} 1 - 4x^{-1/3} &= 0 \\ 4x^{-1/3} &= 1 \\ \frac{1}{\sqrt[3]{x}} &= \frac{1}{4} \\ \sqrt[3]{x} &= 4 \\ x &= 4^3 = 64. \end{aligned}$$

There is a second critical point, though, which would be easy to miss: the derivative

$$f'(x) = 1 - 4x^{-1/3}$$

is undefined when $x = 0$.

Looking at the formula for the derivative, it isn't hard to see that

$$\begin{aligned} f'(x) &> 0 \text{ if } x < 0 \text{ or if } x > 64 \\ f'(x) &< 0 \text{ if } 0 < x < 64. \end{aligned}$$

The critical point at $x = 0$ is therefore a local maximum, and the critical point at $x = 64$ is a local minimum. The global extrema must be either these local extrema or else the endpoints of the interval. Every possible global extremum is therefore listed in the following table.

x	$f(x)$	Nature
-4	$-4 - 12\sqrt{2} \doteq -20.97$	
0	0	Global max.
64	-32	Global min.
125	-25	

The plot of the function is in Figure 4.

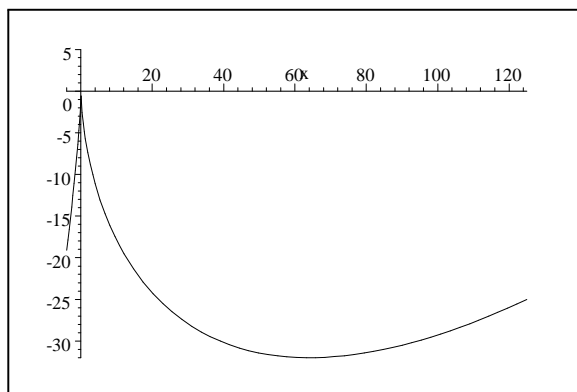


Figure 4: $y = x - 6x^{2/3}$.

4.3.3 Example 3

Finally, let's try the same thing with $f(x) = x^{-2}$ on the interval $(-2, 2)$, i.e., on the open interval $-2 < x < 2$. The derivative is

$$f'(x) = -\frac{2}{x^3},$$

which is never zero, but which is undefined at $x = 0$. There is therefore a critical point at $x = 0$. This critical point is neither a maximum nor a minimum, but a vertical asymptote, as shown in Figure 5.

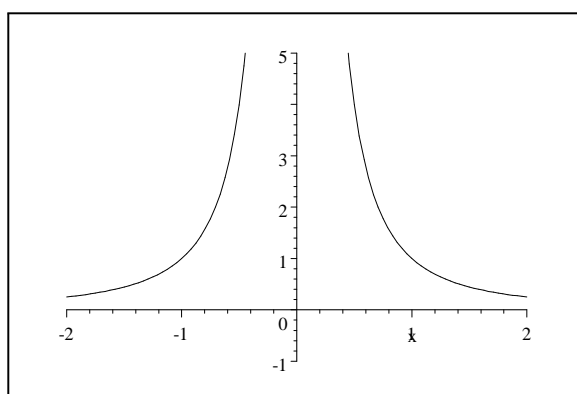


Figure 5: $y = x^{-2}$.

The function $f(x) = x^{-2}$ therefore has no global maximum on the interval $(-2, 2)$. It can be made as large as we like by taking points close to $x = 0$.

What may be more subtle is that $f(x) = x^{-2}$ also has no global minimum on the interval $(-2, 2)$. The value of $f(x)$ gets smaller and smaller as x gets closer and closer to 2 or to -2 . The global minimum therefore can't be at any point except $x = \pm 2$. But the global minimum also can't be at $x = 2$ or $x = -2$, since neither of these points lies in the interval $-2 < x < 2$. All we can therefore say is that $f(x)$ can be made as close as we like to $\frac{1}{4}$ by taking x close enough to one of the endpoints of the interval $(-2, 2)$; but that f has no minimum on this interval.

4.4 Rolle's Theorem

Rolle's Theorem and the Mean Value Theorem are two results which are geometrically obvious, but which play a central role in the formal development of calculus if one is being logically precise. Since we're not trying to prove every last thing we do, these theorems are less central to us; but they are still part of the basic corpus of material people expect you to have learned in calculus.

Rolle's Theorem says that if f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , and if $f(a) = f(b)$, then there is some point c , $a < c < b$, at which $f'(c) = 0$.

The proof is easy: If f is a constant, then $f'(c) = 0$ for every $a < c < b$. If not, then f has to have a global maximum and a global minimum on $[a, b]$. Not both of these can be endpoints of the interval, or else f would be a constant. At least one must therefore be a critical point—a point at which either $f'(c) = 0$ or $f'(c)$ is undefined. Since f is differentiable everywhere on (a, b) , the only critical points are those where $f'(c) = 0$, so we're done.

A typical function for which Rolle's Theorem applies is shown in Figure 6. The points where $f'(c) = 0$ are shown by small segments of the tangent lines at those points.

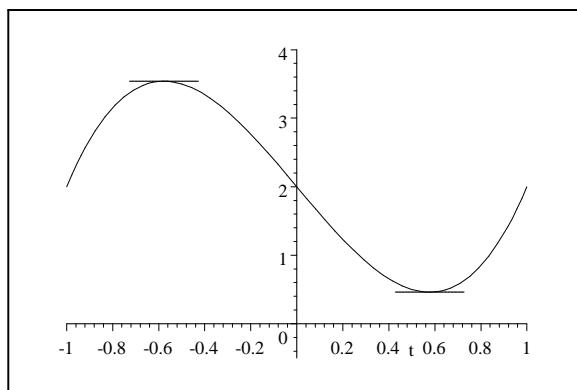


Figure 6: Rolle's Theorem.

The Mean Value Theorem is analogous to Rolle's Theorem, but it drops the condition that $f(a) = f(b)$. It asserts that if f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is some point c , $a < c < b$, at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is actually more intuitive than the notation first makes you think. The right hand side, $\frac{f(b) - f(a)}{b - a}$, is the average rate of change of the function f between $x = a$ and $x = b$. The left hand side, $f'(c)$, is the instantaneous rate of change of f at c . So what the Mean Value Theorem is saying is that there is always a point where the instantaneous rate of change is equal to the average rate of change. If you drive from here to Dayton averaging 60 mph, then at some point along the way, you must have been going exactly 60. Believable?

Another way to think about it is to say that $\frac{f(b) - f(a)}{b - a}$ is the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$. What the Mean Value

Theorem says is that there is a point c between a and b at which the tangent line is parallel to this secant line.

The proof is one of those clever mathematical arguments that uses what we already know in a slick way. Consider the function

$$h(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right].$$

h is just f minus the secant line joining $(a, f(a))$ to $(b, f(b))$. Thus h is a difference of 2 continuous functions, so h is continuous on $[a, b]$. Also,

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

is defined at every point on (a, b) . Finally,

$$\begin{aligned} h(a) &= f(a) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (a - a) + f(a) \right] = 0 \\ h(b) &= f(b) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (b - a) + f(a) \right] = 0. \end{aligned}$$

We can therefore apply Rolle's Theorem to h to get a point c with $a < c < b$ at which

$$\begin{aligned} 0 &= h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \\ f'(c) &= \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

Figure 7 shows a function for which the Mean Value Theorem applies. The secant line joining $(a, f(a))$ to $(b, f(b))$ is also shown, along with the 2 points at which the tangent lines are parallel to this secant line.

4.5 The First Derivative Test

A function f is increasing in the interval $[a, b]$ if for any points $a \leq s < t \leq b$, we have $f(s) < f(t)$. f is decreasing in the interval $[a, b]$ if for any points $a \leq s < t \leq b$, we have $f(s) > f(t)$.

Our intuitive notion has been that increasing functions and positive derivatives go together, and that decreasing functions and negative derivatives go together. We also know we have to be a little more delicate than that: if $f(x) = x^3$, then f is increasing everywhere, but $f'(0) = 3(0^2) = 0$.

The correct theorem turns out to be:

If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

To prove the first case, we prove its contrapositive, which is logically equivalent:

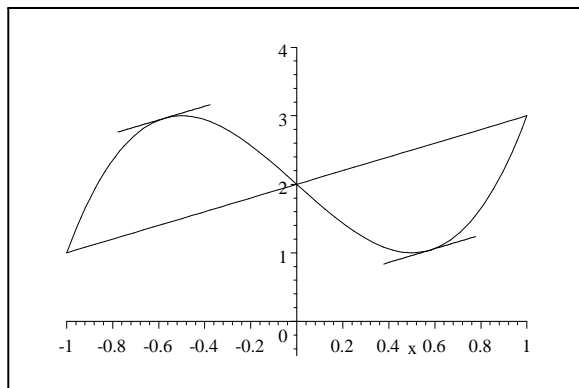


Figure 7: Mean Value Theorem.

If f is not increasing on $[a, b]$, then for some $c \in (a, b)$, $f'(c) \leq 0$.

If f is not increasing on $[a, b]$, then there exist points $a \leq s < t \leq b$ with $f(s) \geq f(t)$. By the Mean Value Theorem, we can therefore find a point c between s and t such that

$$f'(c) = \frac{f(t) - f(s)}{t - s} \leq 0,$$

as claimed. See if you can do the other 2 cases yourself.

The First Derivative Test harvests this idea in order to classify the behavior of f at critical points. It says,

Let f be differentiable in an interval around c , and let $f'(c) = 0$.

If for all $x < c$ near c , $f'(x) < 0$, and for all $x > c$ near c , $f'(x) > 0$, then c is a local minimum for f .

If for all $x < c$ near c , $f'(x) > 0$, and for all $x > c$ near c , $f'(x) < 0$, then c is a local maximum for f .

If for all $x \neq c$ near c , $f'(x) > 0$, or for all $x \neq c$ near c , $f'(x) < 0$, then c is neither a local maximum nor a local minimum for f .

This makes sense because what we are really saying is,

If f is decreasing to the left of c and increasing to the right of c , then c is a local minimum.

If f is increasing to the left of c and decreasing to the right of c , then c is a local maximum.

If f is decreasing on both sides of c or increasing on both sides of c , then c is neither a local minimum nor a local maximum. It's worth taking a minute and drawing pictures of all these situations. Graphs showing three of the things that could happen are shown in Figure 8.

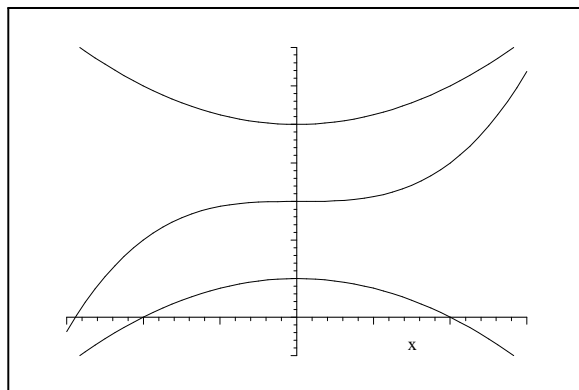


Figure 8: Three functions with $f'(0) = 0$.

4.6 Examples.

4.6.1 Example 1.

Let $f(x) = x^3 - 5x^2 + x - 1$. Find all critical points of f . Determine where f is increasing and decreasing, and locate its local maxima and minima. Sketch the graph of f .

If $f(x) = x^3 - 5x^2 + x - 1$, then $f'(x) = 3x^2 - 10x + 1$. Critical points occur where $f'(x)$ is undefined—which never happens—or where $f'(x) = 0$, i.e., at

$$x = \frac{10 \pm \sqrt{88}}{6} = \frac{5 \pm \sqrt{22}}{3}.$$

It is easy to see that $f'(x)$ is an upward-pointing parabola, which means that

$$\begin{aligned} f'(x) &< 0 \text{ if } \frac{5 - \sqrt{22}}{3} < x < \frac{5 + \sqrt{22}}{3} \\ f'(x) &> 0 \text{ if } x < \frac{5 - \sqrt{22}}{3} \text{ or } x > \frac{5 + \sqrt{22}}{3}. \end{aligned}$$

f therefore has a local maximum at

$$\begin{aligned} \left(\frac{5 - \sqrt{22}}{3}, f\left(\frac{5 - \sqrt{22}}{3}\right) \right) &= \left(\frac{5 - \sqrt{22}}{3}, \frac{-232 + 44\sqrt{22}}{27} \right) \\ &\doteq (0.103194747, -0.948952093) \end{aligned}$$

and a local minimum at

$$\begin{aligned} \left(\frac{5 + \sqrt{22}}{3}, f\left(\frac{5 + \sqrt{22}}{3}\right) \right) &= \left(\frac{5 + \sqrt{22}}{3}, \frac{-232 - 44\sqrt{22}}{27} \right) \\ &\doteq (3.230138587, -16.23623309). \end{aligned}$$

A sketch is in Figure 9.

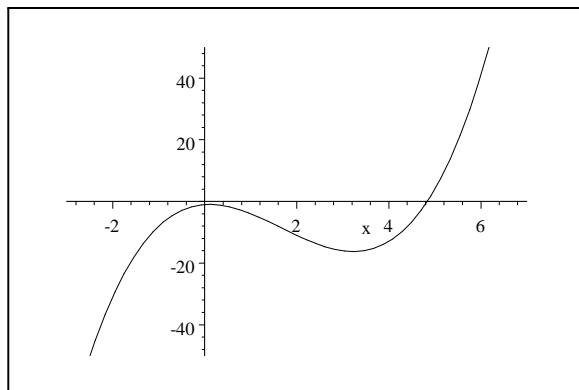


Figure 9: $f(x) = x^3 - 5x^2 + x - 1$.

4.6.2 Example 2.

Let $f(x) = \frac{x^2+2x+2}{x+2}$. Find all critical points of f . Determine where f is increasing and decreasing, and locate its local maxima and minima. Sketch the graph of f .

If $f(x) = \frac{x^2+2x+2}{x+2}$, then

$$\begin{aligned} f'(x) &= \frac{(x+2)(2x+2) - (x^2+2x+2)(1)}{(x+2)^2} \\ &= \frac{x^2+4x+2}{(x+2)^2}. \end{aligned}$$

Critical points occur where $f'(x)$ is undefined (at $x = -2$) or where $f'(x) = 0$, i.e., at

$$x = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}.$$

It is easy to see that the numerator of $f'(x)$ is an upward-pointing parabola, negative between the two roots and positive elsewhere. The denominator of $f'(x)$ is always non-negative. This means that

$$\begin{aligned} f'(x) &< 0 \text{ if } -2 - \sqrt{2} < x < -2 \text{ or if } -2 < x < -2 + \sqrt{2}. \\ f'(x) &> 0 \text{ if } x < -2 - \sqrt{2} \text{ or } x > -2 + \sqrt{2}. \end{aligned}$$

The critical point at $x = -2$ is a vertical asymptote. Because the derivative is negative on both sides of this asymptote, it must be that

$$\begin{aligned} \lim_{x \rightarrow -2^-} f(x) &= -\infty \\ \lim_{x \rightarrow -2^+} f(x) &= +\infty. \end{aligned}$$

f has a local maximum at

$$\begin{aligned} \left(-2 - \sqrt{2}, f\left(-2 - \sqrt{2}\right)\right) &= \left(-2 - \sqrt{2}, -2 - 2\sqrt{2}\right) \\ &\doteq (-3.414213562, -4.828427123) \end{aligned}$$

and a local minimum at

$$\begin{aligned} \left(-2 + \sqrt{2}, f\left(-2 + \sqrt{2}\right)\right) &= \left(-2 + \sqrt{2}, -2 + 2\sqrt{2}\right) \\ &\doteq (-0.585786438, 0.8284271251) \end{aligned}$$

A sketch is in Figure 10.

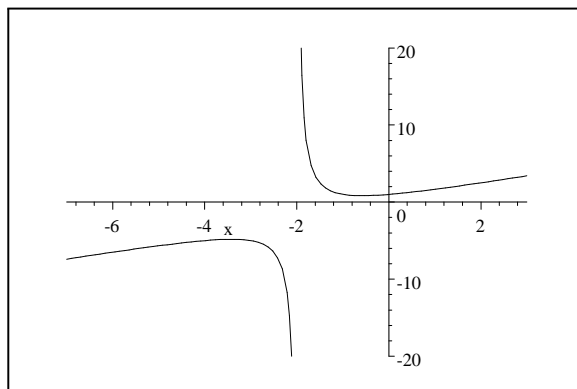


Figure 10: $f(x) = \frac{x^2+2x+2}{x+2}$.

4.6.3 Example 3.

Let $f(x) = 3x^4 - 8x^3 - 1$. Find all critical points of f . Determine where f is increasing and decreasing, and locate its local maxima and minima. Sketch the graph of f .

If $f(x) = 3x^4 - 8x^3 - 1$, then $f'(x) = 12x^3 - 24x^2 = 12x^2(x - 2)$. Critical points occur where $f'(x)$ is undefined—which never happens—or where $f'(x) = 0$, i.e., at

$$x = 0 \text{ or } x = 2.$$

It is easy to see that $f'(x) > 0$ if $x > 2$, and that $f'(x) < 0$ if $x < 2$. This means that f is increasing to the right of 2, and decreasing to the left of 2. f therefore has a local minimum at $(2, f(2)) = (2, 15)$; the critical point at $x = 0$ is neither a local maximum nor a local minimum.

A sketch is in Figure 11.

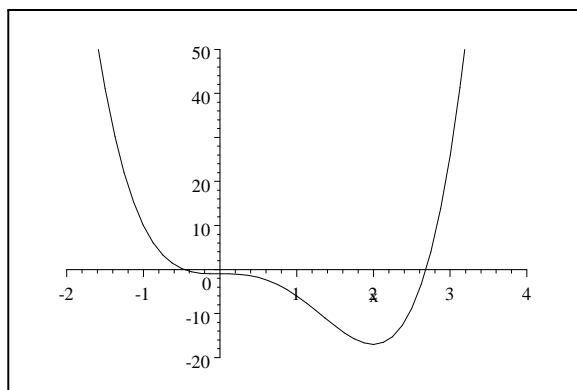


Figure 11: $f(x) = 3x^4 - 8x^3 - 1$.

4.7 A Cautionary history.

We now know that

If for all x in the interval $a < x < b$, $f'(x) > 0$, then f is increasing in the interval $a < x < b$.

If for all x in the interval $a < x < b$, $f'(x) < 0$, then f is decreasing in the interval $a < x < b$.

If for all x in the interval $a < x < b$, $f'(x) = 0$, then f is a constant in the interval $a < x < b$.

So the first derivative gives us a lot of information about the graph of f .

For instance, let $f(x) = 9x^4 - 11x^3 - 3x$. The derivative of f is

$$f'(x) = 36x^3 - 33x^2 - 3 = 3(x - 1)(12x^2 + x + 1).$$

The only root of f' is $x = 1$. It is easy to see that $f'(x) > 0$ if $x > 1$, and that $f'(x) < 0$ if $x < 1$. Thus, f decreases to the point $(1, -5)$, and f then increases to the right of this point. This still leaves a large degree of uncertainty about the shape of the graph of $y = f(x)$, though. Several possibilities are shown in Figures 12, 13, 14, and 15.

All these graphs satisfy the conditions imposed on $f(x) = 9x^4 - 11x^3 - 3x$ by the sign of its first derivative. Can we use calculus to tell which is the right graph?

4.8 Concavity and Second Derivatives.

The answer turns out to be yes, and to rely not on the information the derivative gives us, but on the information the second derivative gives us. To see this, let's apply the first derivative test not to the function f , whose derivative is f' , but to the function f' , whose derivative is f'' . We learn that

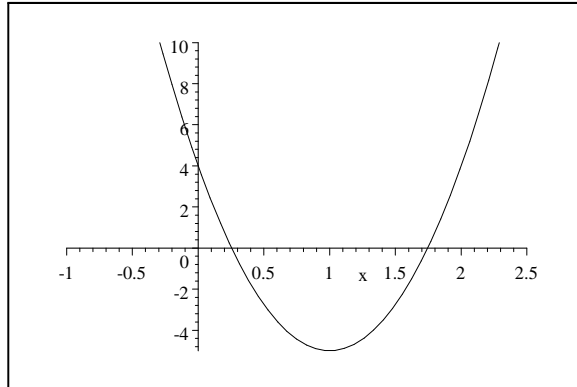


Figure 12: A possible graph of $f(x)$.

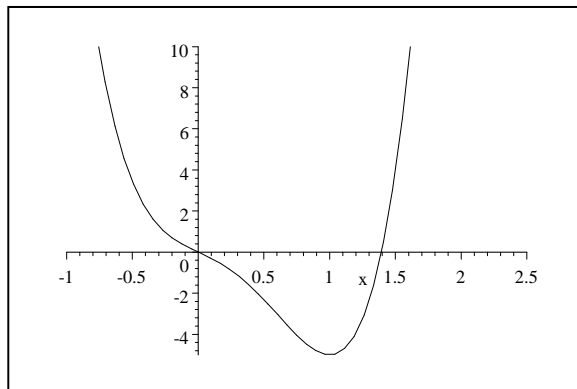


Figure 13: A possible graph of $f(x)$.

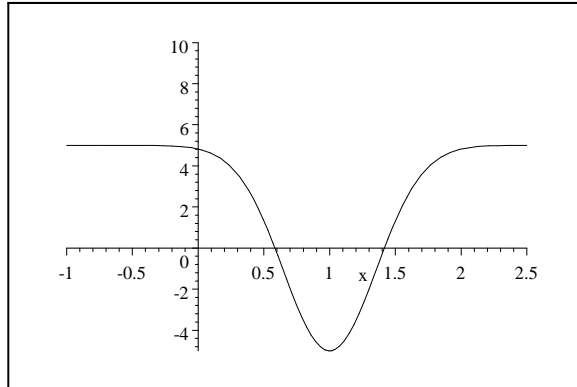


Figure 14: A possible graph of $f(x)$.

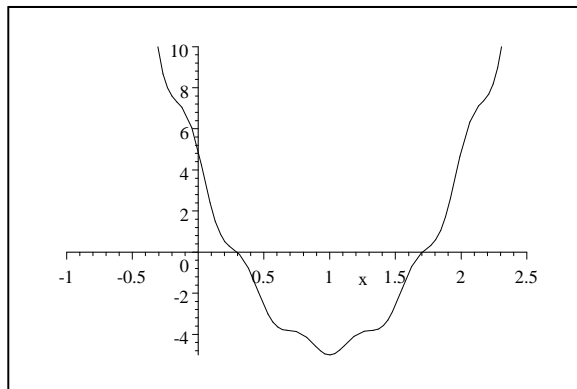


Figure 15: A possible graph of $f(x)$.

If for all x in the interval $a < x < b$, $f''(x) > 0$, then f' is increasing in the interval $a < x < b$.

If for all x in the interval $a < x < b$, $f''(x) < 0$, then f' is decreasing in the interval $a < x < b$.

If for all x in the interval $a < x < b$, $f''(x) = 0$, then f' is a constant in the interval $a < x < b$.

What does it mean for f' to be increasing, though? It means that the slope of the tangent line to f is getting larger, which means that f is either increasing at a faster and faster rate as you go to the right, or that f is decreasing, but doing so more slowly as you go to the right. Functions whose derivatives are increasing are said to be *concave up*. Three functions that are concave up are shown in Figure 16.

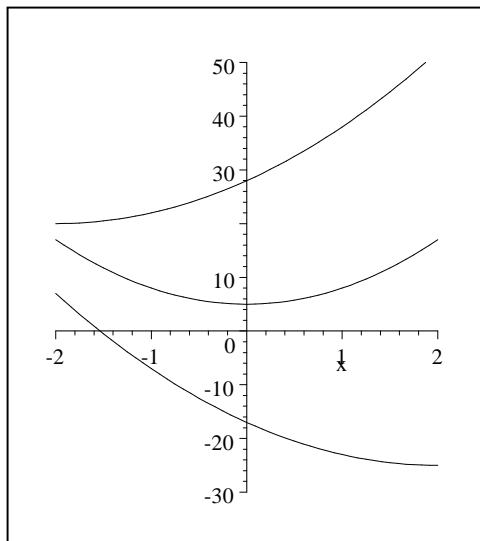


Figure 16: Concave up functions

Similarly, for f' to be decreasing means that the slope of the tangent line to f is getting smaller, which means that f is either decreasing at a faster and faster rate as you go to the right, or that f is increasing, but doing so more slowly as you go to the right. Functions whose derivatives are decreasing are said to be *concave down*. Three concave down functions are shown in Figure 17.

So what we've now learned is

If for all x in the interval $a < x < b$, $f''(x) > 0$, then f is concave up in the interval $a < x < b$.

If for all x in the interval $a < x < b$, $f''(x) < 0$, then f is concave down in the interval $a < x < b$.

If for all x in the interval $a < x < b$, $f''(x) = 0$, then f is a straight line in the interval $a < x < b$.

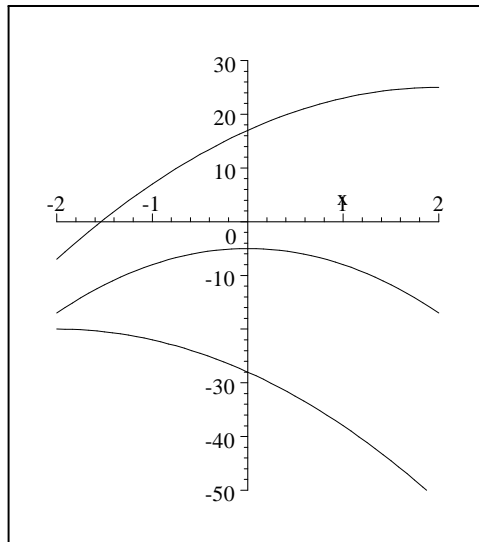


Figure 17: Concave down functions.

One other bit of terminology is useful: An *inflection point* is a point at which the second derivative of f changes sign, i.e., a point c such that f is concave up on one side of c and concave down on the other side of c . The three functions shown in Figure 18 all have inflection points at $x = 0$.

The only way for the second derivative of f to change sign at $x = c$ is if $f''(c)$ is either 0 or undefined; so all inflection points satisfy this condition. It is not the case, however, that every point at which $f''(x)$ is zero or undefined is an inflection point. Consider, for instance, the function $f(x) = x^4$. We have $f'(x) = 4x^3$ and $f''(x) = 12x^2$, so $f''(0) = 0$. But a look at the graph of $y = f(x) = x^4$ (Figure 19) shows that f is concave up everywhere, so that there is no inflection point at $x = 0$. One could also see this algebraically, since clearly $f''(x) = 12x^2 \geq 0$ for all x .

4.9 The cautionary example resolved.

Now let's return to our initial question about the graph of $y = 9x^4 - 11x^3 - 3x$. We have

$$\begin{aligned} y &= 9x^4 - 11x^3 - 3x \\ y' &= 36x^3 - 33x^2 - 3 = 3(x-1)(12x^2 + x + 1) \\ y'' &= 108x^2 - 66x = 6x(18x - 11). \end{aligned}$$

This means that there are possible inflection points for y at $x = 0$ and at $x = \frac{11}{18}$, where $y'' = 0$. In fact, it is easy to see by looking at the sign of each of

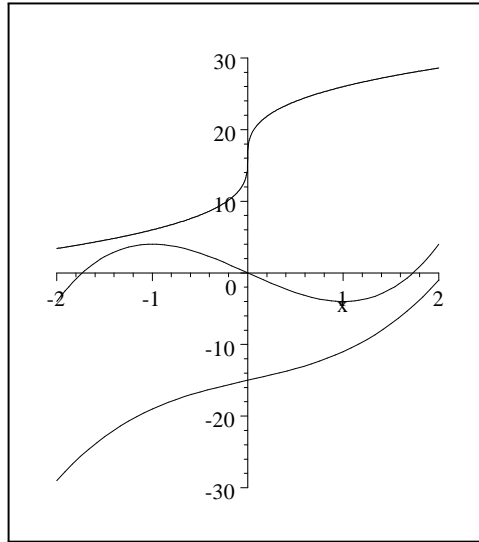


Figure 18: Functions with inflection points at $x = 0$.

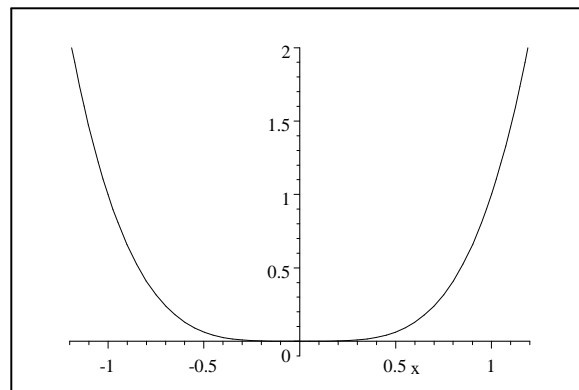


Figure 19: $y = x^4$.

the factors of y'' that

$$\begin{aligned}y'' &> 0 \text{ if } x > \frac{11}{18} \\y'' &< 0 \text{ if } 0 < x < \frac{11}{18} \\y'' &> 0 \text{ if } x < 0.\end{aligned}$$

The function $y = 9x^4 - 11x^3 - 3x$ must therefore be concave down between $x = 0$ and $x = \frac{11}{18}$, and it must be concave up everywhere else. A look at the graphs in Figures 12–15 shows that only the graph in Figure 13 satisfies this condition. It is in fact the correct graph.

4.10 An example of our graphing tools.

Consider the function

$$\begin{aligned}f(x) &= 4x + \frac{1}{x-1} + \frac{1}{x+1} \\f'(x) &= 4 - \frac{1}{(x-1)^2} - \frac{1}{(x+1)^2} = \frac{2(2x^4 - 5x^2 + 1)}{(x-1)^2(x+1)^2} \\f''(x) &= \frac{2}{(x-1)^3} + \frac{2}{(x+1)^3} = \frac{4x(x^2 + 3)}{(x-1)^3(x+1)^3}.\end{aligned}$$

Suppose we want to determine all critical points of f , all inflection points of f , and all extrema of f . We want to know where f is increasing and decreasing, and where it is concave up and concave down. Finally, we want the graph of f .

Since f'' is a bit simpler than f' , or at least, since it seems to factor further, let's start with it. The possible positions of inflection points are at $x = \pm 1$, where $f''(x)$ is undefined, and at $x = 0$, where $f''(x) = 0$. The points at $x = \pm 1$ are not inflection points, but vertical asymptotes, as can quickly be seen by looking at the equation for $f(x)$. On the other hand, it is easy to see by looking at the signs of each of the factors that make up $f''(x)$ that if $-1 < x < 0$, then $f''(x) > 0$, and if $0 < x < 1$, then $f''(x) < 0$. The point $(0, 0)$ is therefore an inflection point for f . It is equally clear that

$$\begin{aligned}f''(x) &< 0 \text{ in } (-\infty, -1) \\f''(x) &> 0 \text{ in } (-1, 0) \\f''(x) &< 0 \text{ in } (0, 1) \\f''(x) &> 0 \text{ in } (1, \infty).\end{aligned}$$

Thus, f is concave down in $(-\infty, -1) \cup (0, 1)$, and concave up in $(-1, 0) \cup (1, \infty)$. Incidentally, it is useful to draw a number line to keep track of these intervals, which lets you keep all your information in one neat picture. A second number line can then hold the information obtained from $f'(x)$.

Now let's look at

$$f'(x) = \frac{2(2x^4 - 5x^2 + 1)}{(x-1)^2(x+1)^2}.$$

There are critical points at $x = \pm 1$, where $f'(x)$ is undefined. These points have already been shown to be vertical asymptotes. There are also critical points at the points where $f'(x) = 0$, i.e., at the roots of

$$2x^4 - 5x^2 + 1 = 0.$$

This equation looks nasty, but in fact, it's just a quadratic equation in x^2 . The quadratic formula says that its roots are

$$\begin{aligned} x^2 &= \frac{5 \pm \sqrt{17}}{4} \\ x &= \pm \sqrt{\frac{5 \pm \sqrt{17}}{4}} \\ x &\doteq \pm 1.51022, \pm 0.468213. \end{aligned}$$

A bit of thought or a bit of plugging in numbers lets us work out the sign of the numerator in $f'(x)$:

$$2(2x^4 - 5x^2 + 1) < 0 \text{ if } 0.468213 < |x| < 1.51022,$$

and $2(2x^4 - 5x^2 + 1) \geq 0$ otherwise. The denominator of $f'(x)$ is always non-negative. So except at the 2 vertical asymptotes, we have

$$\begin{aligned} f'(x) &> 0 \text{ if } x < -1.51 \\ f'(x) &< 0 \text{ if } -1.51 < x < -0.47 \\ f'(x) &> 0 \text{ if } -0.47 < x < 0.47 \\ f'(x) &< 0 \text{ if } 0.47 < x < 1.51 \\ f'(x) &> 0 \text{ if } 1.51 < x. \end{aligned}$$

The critical points at -1.51 and 0.47 are therefore local maxima, while the ones at -0.47 and 1.51 are local minima. When we take all this information together, we get the graph of $y = f(x)$, which is shown in Figure 20.

4.11 The second derivative test.

Although we can use the second derivative of f like this to get complicated information about the graph of f , there is also a simple thing we can do with the second derivative. Suppose $f'(c) = 0$. Then $x = c$ might be a local maximum or a local minimum or neither. One way to tell in most cases is to take the second derivative of f . If $f'(c) = 0$, and if $f''(x)$ exists and is continuous in a neighborhood of c , then

$$\begin{aligned} \text{If } f''(c) &> 0, \text{ then } (c, f(c)) \text{ is a local minimum.} \\ \text{If } f''(c) &< 0, \text{ then } (c, f(c)) \text{ is a local maximum.} \\ \text{If } f''(c) &= 0, \text{ then anything can happen at } (c, f(c)). \end{aligned}$$

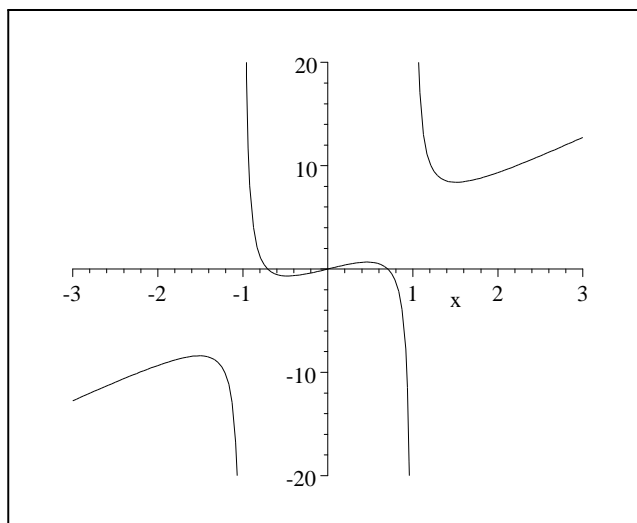


Figure 20: $f(x) = 4x + \frac{1}{x-1} + \frac{1}{x+1}$.

Calculus books typically prove this, but it also just makes sense. Try to draw sketches, and you'll see that the only way for f to have a horizontal tangent line at $(c, f(c))$ and for f to be concave up at $(c, f(c))$ is if $(c, f(c))$ is a local minimum. Similarly, the only way for f to have a horizontal tangent line at $(c, f(c))$ and for f to be concave down at $(c, f(c))$ is if $(c, f(c))$ is a local maximum.

To see that anything can happen if $f'(c) = 0$ and $f''(c) = 0$, consider the functions

$$\begin{aligned} f(x) &= x^4 \\ g(x) &= -x^4 \\ h(x) &= x^3 \\ k(x) &= -x^3. \end{aligned}$$

All these functions have $y' = y'' = 0$ at $x = 0$. But f has a local and global minimum at $x = 0$, g has a local and global maximum at $x = 0$, h has an inflection point at $x = 0$ and is always increasing, and k has an inflection point at $x = 0$ and is always decreasing.

4.11.1 Two quick examples.

If $f(x) = 3x^4 + 4x^3 - 12x^2 + 13$, then $f'(x) = 12x^3 + 12x^2 - 24x = 12x(x-1)(x+2)$. The function f therefore has critical points at 0, 1, and -2 . The second derivative is $f''(x) = 36x^2 + 24x - 24$, so $f''(0) = -24 < 0$, $f''(1) = 36 > 0$, $f''(-2) =$

$72 > 0$. Therefore f has a local maximum at $(0, 13)$, a local minimum at $(1, 8)$, and a local minimum at $(-2, -19)$. The plot is shown in Figure 21.

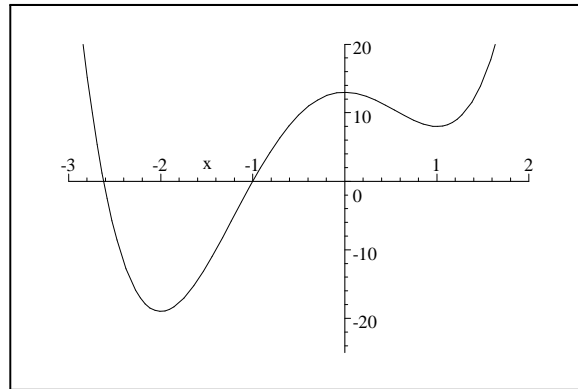


Figure 21: $y = 3x^4 + 4x^3 - 12x^2 + 13$.

If $g(x) = 4x^5 + 5x^4 + 11$, then $g'(x) = 20x^4 + 20x^3 = 20x^3(x + 1)$. The function g therefore has critical points at 0 and -1 . The second derivative is $g''(x) = 80x^3 + 60x^2$, so $g''(0) = 0$ and $g''(-1) = -20 < 0$. Therefore g has a local maximum at $(-1, 12)$. The second derivative test doesn't tell us the nature of the critical point at $x = 0$, but by looking at $g'(x) = 20x^3(x + 1)$, it is clear that $g'(x) < 0$ if x is slightly less than 0, and that $g'(x) > 0$ if x is slightly greater than 0. Thus, $(0, 11)$ must be a local minimum for g . The plot is shown in Figure 22.

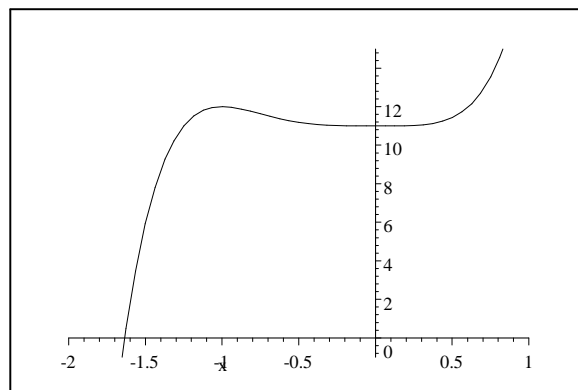


Figure 22: $y = 4x^5 + 5x^4 + 11$

4.12 Limits at Infinity.

By using the information the first and second derivatives of f give us, we can come very close to completely plotting the graphs of most functions. There is only one bit of the picture we haven't yet thought about—the behavior of functions at large (or large negative) values of x .

Consider, for instance, the function

$$\begin{aligned}f(x) &= \frac{x^2 - 1}{x} \\f'(x) &= \frac{x^2 + 1}{x^2} \\f''(x) &= -\frac{2}{x^3}.\end{aligned}$$

This function has no critical points or inflection points, since neither the first nor second derivatives can equal 0. It has a vertical asymptote at $x = 0$, and

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{x^2 - 1}{x} = -\infty \\ \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{x^2 - 1}{x} = +\infty.\end{aligned}$$

Both the numerator and denominator of the derivative are always non-negative; so except at the point $x = 0$, we have $f'(x) > 0$. The function f is therefore increasing except at the discontinuity at the vertical asymptote. Just looking at the second derivative, we can see that

$$\begin{aligned}f''(x) &< 0 \text{ if } x > 0, \\ f''(x) &> 0 \text{ if } x < 0.\end{aligned}$$

Thus f is concave down for positive x , and concave up for negative x .

Even with all this information, though, there are still a variety of different shapes the graph of f could take. Three possibilities are shown in Figures 23, 24, and 25.

These plots differ in the behavior of f for large values of x and for large negative values of x . How can we tell which of these graphs is right?

Part of the answer is that the graphs in Figures 23–25 differ in their limits at infinity. For the function in Figure 23,

$$\lim_{x \rightarrow \infty} f(x) = 0,$$

while for the functions in Figures 24 and 25,

$$\lim_{x \rightarrow \infty} f(x) = \infty.$$

So how would we work out

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x}?$$

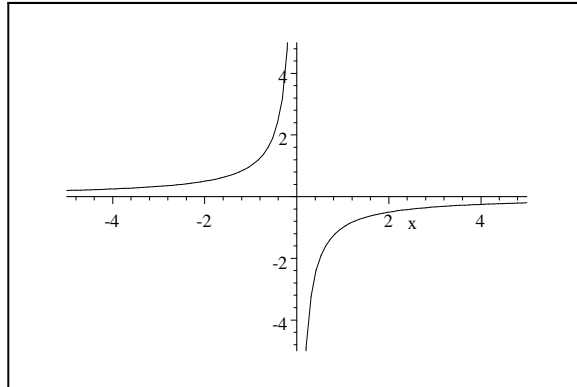


Figure 23: A possible graph for $f(x)$.

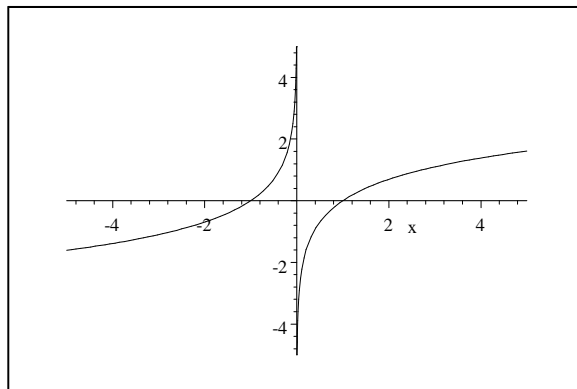


Figure 24: A possible graph for $f(x)$.

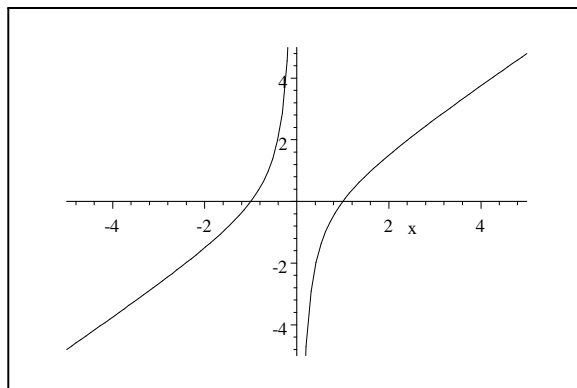


Figure 25: A possible graph for $f(x)$.

Well, the natural thing to do would be to break the function up as

$$f(x) = \frac{x^2 - 1}{x} = x - \frac{1}{x}.$$

When x gets very large, $\frac{1}{x} \rightarrow 0$; so $f(x) = x - \frac{1}{x} \doteq x \rightarrow \infty$. Thus,

$$\lim_{x \rightarrow \infty} f(x) = \infty.$$

This eliminates the graph in Figure 23 from consideration, and leaves us with something like the graphs in Figures 24 and 25.

Now, this example relied on our seeing how to break $f(x)$ up into a piece that went to 0 and a piece that obviously went to ∞ . We may not always find this trivial. How, in general, can we compute limits of rational functions at $\pm\infty$? To find out, let's just look at 3 typical examples.

First, here's a case where the degree of the denominator is larger than the degree of the numerator. In this case, we divide both top and bottom by the highest power of x found in the denominator.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x^3 - 11x^2 + 5}{4x^5 + x^2 - 3x} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^3}{x^5} - \frac{11x^2}{x^5} + \frac{5}{x^5}}{\frac{4x^5}{x^5} + \frac{x^2}{x^5} - \frac{3x}{x^5}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{2}{x^2} - \frac{11}{x^3} + \frac{5}{x^5}}{4 + \frac{1}{x^3} - \frac{3}{x^4}}. \end{aligned}$$

As $x \rightarrow \infty$, the numerator approaches 0 and the denominator approaches 4; so the limit must be $\frac{0}{4} = 0$. It isn't hard to see that the same thing would happen whenever the denominator has higher degree than the numerator.

Next, let's look at a case where the numerator and denominator have the same degree. Again, we divide top and bottom by the highest power of x found

in the denominator.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x^5 - 11x^2 + 5}{4x^5 + x^2 - 3x} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^5}{x^5} - \frac{11x^2}{x^5} + \frac{5}{x^5}}{\frac{4x^5}{x^5} + \frac{x^2}{x^5} - \frac{3x}{x^5}} \\ &= \lim_{x \rightarrow \infty} \frac{2 - \frac{11}{x^3} + \frac{5}{x^5}}{4 + \frac{1}{x^3} - \frac{3}{x^4}}.\end{aligned}$$

As $x \rightarrow \infty$, the numerator approaches 2 and the denominator approaches 4; so the limit must be $\frac{2}{4} = \frac{1}{2}$. It isn't hard to see that this is also typical—if the numerator and denominator have the same degree, then the limit is the ratio of their leading coefficients.

Finally, here's an example where the numerator has higher degree than the denominator. This case takes a bit more thought. Let

$$f(x) = \frac{2x^5 - 4x^4 + x^3 + 4x^2 - 3x + 5}{x^3 - 2x^2 + 8}.$$

If we again divide top and bottom by the highest power of x in the denominator, we get something like this:

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{2x^5 - 4x^4 + x^3 + 4x^2 - 3x + 5}{x^3 - 2x^2 + 8} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^5}{x^3} - \frac{4x^4}{x^3} + \frac{x^3}{x^3} + \frac{4x^2}{x^3} - \frac{3x}{x^3} + \frac{5}{x^3}}{\frac{x^3}{x^3} - \frac{2x^2}{x^3} + \frac{8}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{2x^2 - 4x + 1 + \frac{4}{x} - \frac{3}{x^2} + \frac{5}{x^3}}{1 - \frac{2}{x} + \frac{8}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 \left(2 - \frac{4}{x} + \frac{1}{x^2} + \frac{4}{x^3} - \frac{3}{x^4} + \frac{5}{x^5}\right)}{1 - \frac{2}{x} + \frac{8}{x^3}}.\end{aligned}$$

Now one reasons like this: When x is very large in absolute value, the denominator here is close to 1. The numerator is x^2 times something very close to 2. So roughly, the whole function is $\frac{2x^2}{1}$, which approaches $+\infty$ when $x \rightarrow \pm\infty$.

Now, calculus books seem not to want to do this, but it's very easy at this point to be a bit more sophisticated than this. If we take the original rational function and just divide the denominator into the numerator with long division, we get

$$\frac{2x^5 - 4x^4 + x^3 + 4x^2 - 3x + 5}{x^3 - 2x^2 + 8} = 2x^2 + 1 + \frac{-3x + 2}{x^3 - 2x^2 + 8}.$$

When x is large in absolute value, the fraction $\frac{-3x+2}{x^3-2x^2+8}$ is close to 0, since its denominator has a bigger degree than its numerator. Thus, for large x ,

$$f(x) \doteq 2x^2 + 1.$$

It is therefore the case that

$$\lim_{x \rightarrow \pm\infty} \frac{2x^5 - 4x^4 + x^3 + 4x^2 - 3x + 5}{x^3 - 2x^2 + 8} = \lim_{x \rightarrow \infty} (2x^2 + 1) = +\infty;$$

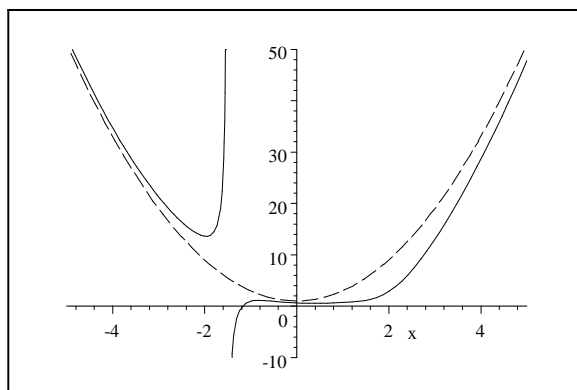


Figure 26: A function and its asymptote.

but one can say more than that. The function $f(x)$ is asymptotic to $2x^2 + 1$. Figure 26 shows the graphs of $f(x)$ and $2x^2 + 1$ together.

Let's finish this section with the example we started with: $f(x) = \frac{x^2-1}{x}$. If we do just what we were asked to do in this last example and divide through by x , we get $f(x) = x - \frac{1}{x}$. When x is large in absolute value, $\frac{1}{x}$ is close to 0; so $f(x) \doteq x$. Figure 27 shows x and $f(x)$ together, confirming that the two curves are asymptotic, and and showing that the plot in Figure 25 was correct.

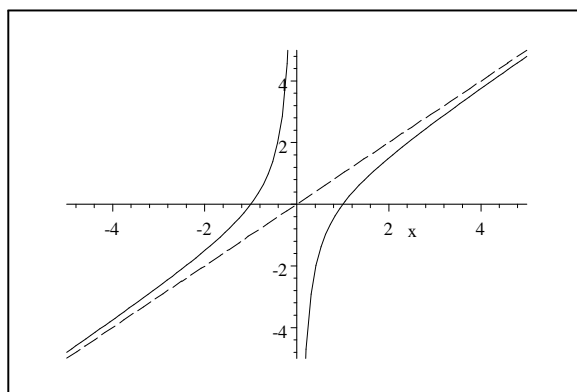


Figure 27: $y = \frac{x^2-1}{x}$ and $y = x$.

4.13 A sketchy summary.

For me, a useful way to conceptualize much of what we have done so far with derivatives and graphing is as follows:

Let f and f' and f'' be continuous in an interval. This implies that if $f'(c) > 0$, then $f'(x) > 0$ for every x in a neighborhood of c . The same sort of thing holds for f and f'' , and if the function is less than 0 instead of greater. Functions like this are the normal sort of unsurprising functions. There are then three possible kinds of behavior at a point x .

If $f'(x) > 0$, then f is increasing near x .

If $f'(x) < 0$, then f is decreasing near x .

If $f'(x) = 0$, then we don't know what's happening at x . There could be a local maximum or minimum the way $-x^2$ and x^2 , resp., have at $x = 0$. The function f could also be increasing or decreasing near x , the way x^3 and $-x^3$, resp., are at $x = 0$. It's interesting that these points where there are lots of possible kinds of behavior, and where we don't know what's going on, are just the points that interest us most.

The second derivative has a similar set of possibilities.

If $f''(x) > 0$, then f is concave up near x .

If $f''(x) < 0$, then f is concave down near x .

If $f''(x) = 0$, then we don't know what's happening at x . There could be an inflection point like $x^3 + x$ has at $x = 0$. There could also be a maximum or minimum at x , as $-x^4$ and x^4 , resp., have at $x = 0$.

Finally, if $f'(x) = 0$, then there are again three possibilities depending on the sign of the second derivative at x .

If $f''(x) > 0$, then f has a local minimum at x .

If $f''(x) < 0$, then f has a local maximum at x .

If $f''(x) = 0$, then we don't know what's happening at x . There could be an inflection point like x^3 has at $x = 0$. There could also be a maximum or minimum at x , as $-x^4$ and x^4 , resp., have at $x = 0$.

It seems to me to be a useful point of view, as well as one the calculus books don't often lead one to understand, that conditions like $f'(x) = 0$ or $f''(x) = 0$ are not telling you that x is an extremum or an inflection point. Instead, they are telling you that x is one of the few points whose status we don't know, and which we ought therefore to investigate more fully.

4.14 Max-Min Problems.

One of the big things calculus is good for is solving problems in which one looks for the maximum or minimum value of some quantity. The tools that let us plot graphs are, after all, exactly the tools one needs to find global or local maxima and minima. In most of these problems, there is a common method: Find the quantity to be optimized. Write that quantity as a function of a single variable. Find the maximum or minimum of that function. Here are a few examples, taken from the appallingly standardized list of standard optimization problems one meets in this business.

4.14.1 Maximizing area.

A typical version of this problem is something like this: A farmer wants to fence in a corral, which is supposed to be a rectangle divided down the center by another fence, as shown in Figure 28. (Presumably the farmer has an infinite amount of flat land, and he is raising 2 types of animals that don't play well together, like sheep and jaguars.) The total amount of fencing available for this project is 200 yards. What are the dimensions of the corral with the maximum area?

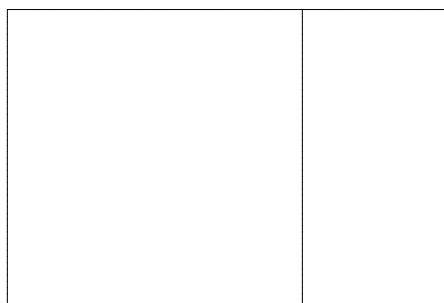


Figure 28: Corral.

If we let x and y be the dimensions of the corral, then we know the quantity we are trying to maximize is the area, $A = xy$. This isn't yet a function of 1 variable, though—we need to get rid either of x or of y by finding some equation relating the two.

This equation is the constraint that $2x + 3y = 200$, which says that the total amount of fencing is 200 yards. Using this, we can write

$$y = \frac{200 - 2x}{3},$$

which means that the function we want to maximize is

$$A(x) = x \left(\frac{200 - 2x}{3} \right) = \frac{200}{3}x - \frac{2}{3}x^2.$$

We want the maximum value of this function on the interval $0 \leq x \leq 100$, since clearly x can't be less than 0 and since if $x > 100$, then $y = \frac{200-2x}{3} < 0$.

At this point, the problem is easy: we compute

$$A'(x) = \frac{200}{3} - \frac{4}{3}x.$$

The only critical point happens when $A'(x) = 0$, at $x = 50$. The global maximum area must therefore be either at this point or at one of the endpoints of

the interval $[0, 100]$. Since the area is clearly 0 at the endpoints, though, the maximum must be when $x = 50$, $y = \frac{100}{3}$, $A = \frac{5000}{3}$.

4.14.2 Minimizing length.

A variation on the corral problem would be this: Suppose the farmer needs the same shape corral as in the previous problem. He knows, however, that the corral needs a total area of 4000 square yards. What's the minimum amount of fencing needed to do the job, and what are the dimensions of the resulting corral?

This time, the quantity to be minimized is the total fence length, $2x + 3y$. As before, this isn't a function of 1 variable, so we can't yet use calculus on it.

We can get rid of one of the two variables, though, by remembering that the area is supposed to be $xy = 4000$ square yards, which means that $y = \frac{4000}{x}$. Plugging this in for y means that the quantity to be minimized is

$$l(x) = 2x + \frac{12000}{x}.$$

We need to minimize this on the interval $(0, +\infty)$, since the width can't be less than 0, but the length can be as long as desired. In real life, one's neighbors might not be so obliging.

As before, the problem is really pretty easy by now. The function $l(x)$ has a single critical point where

$$0 = l'(x) = 2 - \frac{12000}{x^2},$$

i.e., where $x^2 = 6000$, or $x = \sqrt{6000} = 20\sqrt{15} \doteq 77.5$ yards. It is clear that if x is close to infinity, then the amount of fencing needed is very large. This is also true when x is close to 0, since in that case, y is very large. The global minimum must therefore be at the critical point, where $x = 20\sqrt{15}$, $y = 4000/x = \frac{40}{3}\sqrt{15}$, $l = 80\sqrt{15}$.

4.14.3 Minimizing surface area.

A more obviously useful problem is the following: We want to construct a cylindrical barrel designed to hold 8 cubic feet of fluid. What are the dimensions that minimize the amount of steel used for the top, bottom, and sides of the barrel?

If we imagine that the barrel is h feet high and has a radius of r feet, then the amount of steel used for each of the top and bottom is πr^2 . The amount used for the side is $2\pi r h$. So we are trying to minimize the surface area

$$S = \pi r^2 + 2\pi r h.$$

As usual, this isn't a function of one variable, but of two, r and h . We need an equation relating these two variables in order to get rid of one. This equation

is the constraint that the volume of the barrel be 8 cubic feet, which says in algebra that

$$\begin{aligned}\pi r^2 h &= 8 \\ h &= \frac{8}{\pi r^2}.\end{aligned}$$

This means we need to minimize

$$S(r) = \pi r^2 + 2\pi r \frac{8}{\pi r^2} = \pi r^2 + \frac{16}{r}.$$

We presumably want to do this on the interval $0 < r < +\infty$, though the barrels built near the ends of this interval would not be easy to handle.

Once we're down to calculus, it's not hard. It's clear that if r is close either to 0 or to ∞ , then $S(r)$ is very large. The minimum therefore has to happen at a critical point of S . Now,

$$S'(r) = 2\pi r - \frac{16}{r^2} = \frac{2\pi r^3 - 16}{r^2},$$

so the only critical point for S in the interval $(0, \infty)$ occurs where

$$\begin{aligned}2\pi r^3 - 16 &= 0 \\ r^3 &= \frac{8}{\pi} \\ r &= \frac{2}{\pi^{1/3}}.\end{aligned}$$

At this point,

$$h = \frac{8}{\pi r^2} = \frac{2}{\pi^{1/3}}.$$

It's worthwhile to contemplate why it might be that real barrels are not built to dimensions like these.

4.14.4 Minimizing time.

Finally, let's do a problem which is posed in a frivolous way, but which leads us to some very important principles.

A fisherman is 1 mile out in a lake with a straight east-west shoreline when he catches a stupendous fish. He wants to show this fish to his friends as soon as possible. The friends are at a bar 1 mile inland and 2 miles west of his position, as shown in Figure 29. The fisherman can row at 2 mph, and can run at 5 mph. To what point should he row in order to minimize his travel time to the bar?

The optimal path will probably not be a straight line, since the fisherman can run faster than he can row. He will therefore probably want to head more nearly toward the shore, in order to trade rowing distance for running distance.

As shown in the figure, if the fisherman heads for a point a distance x from the nearest point on shore, then by the Pythagorean Theorem, he needs to row

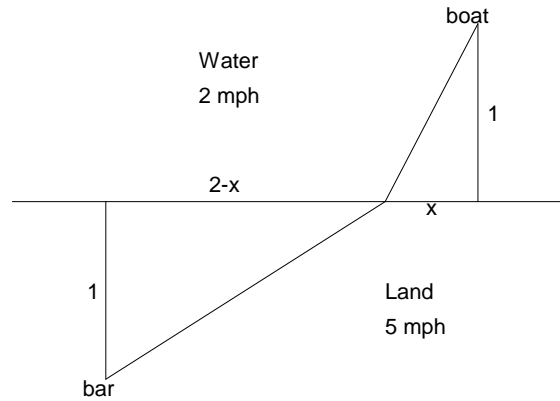


Figure 29: The race to the bar.

a distance of $\sqrt{1+x^2}$ and to run a distance $\sqrt{1+(2-x)^2}$. The time this will take is

$$T(x) = \frac{\sqrt{1+x^2}}{2} + \frac{\sqrt{1+(2-x)^2}}{5}.$$

To minimize the time, we first take its derivative and look for critical points. We find that

$$\begin{aligned} T'(x) &= \frac{2x}{4\sqrt{1+x^2}} + \frac{-2(2-x)}{10\sqrt{1+(2-x)^2}} \\ &= \frac{5x\sqrt{1+(2-x)^2} + (2x-4)\sqrt{1+x^2}}{10\sqrt{1+x^2}\sqrt{1+(2-x)^2}}. \end{aligned}$$

This looks bad, but at least the denominator is always positive, so that $T'(x)$ is always defined. We therefore need only find where $T'(x) = 0$. This happens when the numerator is 0, i.e., when

$$\begin{aligned} 5x\sqrt{1+(2-x)^2} &= -(2x-4)\sqrt{1+x^2} \\ 25x^2(1+(2-x)^2) &= (2x-4)^2(1+x^2) \\ 21x^4 - 84x^3 + 105x^2 + 16x - 16 &= 0. \end{aligned}$$

You probably can't solve this equation exactly—I can't, anyway—but you can use Maple to get approximate solutions, which are roughly

$$\begin{aligned} x &= -0.3972 \\ x &= 0.3632. \end{aligned}$$

Clearly the second of these is the one we want. The fisherman obviously shouldn't row away from the bar.

Now, there is another and deeper way to think about this problem. Let θ_1 and θ_2 be the angles marked in Figure 30 below, and let v_1 and v_2 be the velocities at which the fisherman can move in water and on land, resp.

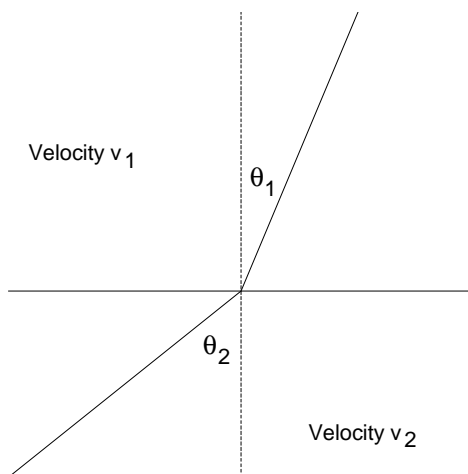


Figure 30: Snell's Law.

Then $T'(x) = 0$ is equivalent to

$$\begin{aligned} \frac{2x}{4\sqrt{1+x^2}} &= \frac{2(2-x)}{10\sqrt{1+(2-x)^2}} \\ \frac{x}{2\sqrt{1+x^2}} &= \frac{(2-x)}{5\sqrt{1+(2-x)^2}} \\ \frac{\sin \theta_1}{v_1} &= \frac{\sin \theta_2}{v_2}. \end{aligned}$$

This last equation is called Snell's law, and applies not just to fishermen, but to the motion of light through two different media in which light travels at different speeds. When a ray of light traveling through air enters water, it bends at an angle dictated by Snell's Law, where v_1 and v_2 are the speeds of light in air and in water, resp. Thus, light travels just like the fisherman, always taking the path that minimizes its travel time. This is true even if the ray of light is traveling through a medium in which its velocity—called the index of refraction—varies continuously. There is a whole branch of mathematics called the calculus of variations which is aimed at solving problems of determining the optimum path in situations like this, and it turns out that what are called variational principles like Snell's Law play a big role in physics in a variety of

situations. If you want to learn more about this, Chapter 19 of Volume 2 of the Feynman Lectures is devoted to this idea. It's an almost verbatim lecture of Feynman's and it's a lovely thing to read for anybody who thinks this sort of math and physics is exciting. You saw it here first, though.