

## 5 Antiderivatives, Integration, and the Fundamental Theorem of Calculus

If you remember back to the beginning of the class, we said that calculus was about 2 problems. One problem was finding slopes of curves; we have certainly resolved this problem completely. The second problem was finding the areas under curves. We haven't thought a moment about this second problem since the first week of class. We ought to come back and see what we can do about it.

We conjectured at the beginning of the semester that the problem of computing slopes and the problem of computing areas seemed to be inverses of one another. That is, if  $g(x)$  was the slope of  $f(x)$  (what we have now learned to call  $f'(x)$ ) then the area under the curve  $g(x)$  seemed to be  $f(x)$ .

There are a lot of details to be worked out before we can even be completely clear exactly what claim is being made here. But it is clear that we need to think about computing areas, and that if we want to make progress on our conjecture, then we should think about the problem of finding  $f(x)$  if we are given  $f'(x)$ .

### 5.1 Antiderivatives.

If  $f(x) = F'(x)$ , so that the function  $f(x)$  is the derivative of  $F(x)$ , then the function  $F(x)$  is called an antiderivative of  $f(x)$ , or an indefinite integral of  $f(x)$ . For instance, if we have

$$\begin{aligned}F(x) &= x^3 + \sin x + 2 \\f(x) &= 3x^2 + \cos x\end{aligned}$$

then  $f$  is the derivative of  $F$ , and  $F$  is an antiderivative of  $f$ . It is right to say "an antiderivative," not "the antiderivative," since  $f(x)$  has lots of antiderivatives, among them

$$\begin{aligned}F(x) &= x^3 + \sin x + 2 \\G(x) &= x^3 + \sin x - 3 \\H(x) &= x^3 + \sin x \\K(x) &= x^3 + \sin x - 17\pi^2.\end{aligned}$$

Clearly all these functions have the same derivative,  $f(x) = 3x^2 + \cos x$ . It isn't hard to prove the following

**Theorem:** If  $F(x)$  is any antiderivative of a continuous function  $f(x)$ , then  $G(x)$  is an antiderivative of  $f(x)$  if and only if  $G(x) = F(x) + c$  for some constant  $c$ .

**Proof:** The hypothesis is that  $F'(x) = f(x)$ . From this it follows immediately that if  $G(x) = F(x) + c$ , then  $G'(x) = F'(x) = f(x)$ , so that  $G$  is also an antiderivative of  $f$ .

Conversely, suppose  $G(x)$  is an antiderivative of  $f(x)$ , so that  $G'(x) = f(x)$ . Then the function  $H(x) = G(x) - F(x)$  satisfies

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0.$$

If  $H(x)$  is not a constant, then we can find 2 points  $a$  and  $b$  at which  $H(a) \neq H(b)$ . The Mean Value Theorem therefore says that for some point  $s$  between  $a$  and  $b$ ,

$$H'(s) = \frac{H(b) - H(a)}{b - a} \neq 0,$$

contradicting the fact that  $H'(x)$  is everywhere zero. Thus,  $H(x) = c$  for some constant  $c$ ; so  $G(x) = F(x) + c$ , as claimed.

A generic antiderivative for  $f(x)$  is denoted, for reasons that will eventually become clear, as

$$F(x) = \int f(x) dx.$$

Thus, the example we've been looking at above could be written as

$$\int (3x^2 + \cos x) dx = x^3 + \sin x + c,$$

the  $+c$  indicating that any constant can be added to  $x^3 + \sin x$  to produce a valid antiderivative.

Since antiderivatives and derivatives undo one another, we have

$$\begin{aligned} \int f'(x) dx &= f(x) + c. \\ \frac{d}{dx} \left( \int f(x) dx \right) &= f(x). \end{aligned}$$

### 5.1.1 Formulas for antiderivatives.

Every formula for derivatives is also a formula for antiderivatives. Thus, the formula

$$\frac{d}{dx} (x^3) = 3x^2$$

is equivalent to

$$\int 3x^2 dx = x^3 + c,$$

and

$$\frac{d}{dx} (\sin x) = \cos x$$

is equivalent to

$$\int \cos x dx = \sin x + c.$$

All our work on derivatives has therefore left us with a huge collection of integration formulas. How can we organize everything we know about antiderivatives?

We might begin with the integral equivalent to the formula  $\frac{d}{dx}x^n = nx^{n-1}$ . We could write this as

$$\int nx^{n-1} dx = x^n + c,$$

but a more useful formula might be replace  $n$  by  $n + 1$  and then to divide by  $n + 1$  to get

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c.$$

One can rationalize this informally by saying that to differentiate  $x^n$ , one decreases the exponent  $n$  by 1 and then multiplies by the old exponent  $n$ . Therefore to take an antiderivative of  $x^n$ , which is the reverse operation, one ought to increase the exponent by 1 and then divide by the new exponent  $n + 1$ . More formally, one can verify the integral formula as one always does: the integral formula is equivalent to

$$\frac{d}{dx} \left( \frac{x^{n+1}}{n+1} \right) = x^n,$$

which can be verified just by using our differentiation formulas.

This idea is worth underscoring. It may not always be easy—and indeed, it is not always possible—to find a formula for the antiderivative of a function  $f(x)$ . But any conjectured integration formula can easily be checked by taking the derivative of the purported integral, and verifying that its derivative is the original function.

Here are a few other basic formulas for indefinite integrals.

$$\begin{aligned} \int \sin x dx &= -\cos x + c \\ \int \cos x dx &= \sin x + c \\ \int 1 dx &= x + c \\ \int \sec^2 x dx &= \tan x + c \\ \int \sec x \tan x dx &= \sec x + c. \end{aligned}$$

The last two of these formulas are the beginning of a more general situation: integral formulas often arise as accidental consequences of calculations involving derivatives.

It is worth noting two general formulas that let one combine these basic facts and compute more antiderivatives.

$$\begin{aligned} \int kf(x) dx &= k \int f(x) dx \\ \int (f(x) \pm g(x)) dx &= \int f(x) dx \pm \int g(x) dx. \end{aligned}$$

As usual, it's easy to prove these integration formulas by turning them into differentiation formulas. If

$$\int f(x) dx = F(x) + c,$$

then  $F'(x) = f(x)$ . The formula

$$\frac{d}{dx} (kF(x)) = kF'(x) = kf(x)$$

can then be rewritten as

$$\int kf(x) dx = kF(x) + c = k \int f(x) dx,$$

proving the first formula above. For the second, suppose that

$$\begin{aligned} \int f(x) dx &= F(x) + c \\ \int g(x) dx &= G(x) + c. \end{aligned}$$

These are equivalent to  $F'(x) = f(x)$  and  $G'(x) = g(x)$ . The differentiation formula

$$\frac{d}{dx} (F(x) \pm G(x)) = F'(x) \pm G'(x) = f(x) \pm g(x)$$

can then be written as

$$\int (f(x) \pm g(x)) dx = F(x) \pm G(x) + c = \int f(x) dx + \int g(x) dx.$$

### 5.1.2 Three harder formulas.

If we're serious in saying that every differentiation formula gives rise to an integral formula, then we ought at least to look briefly at the fanciest differentiation formulas we've developed, the Product, Quotient, and Chain Rules,

$$\begin{aligned} \frac{d}{dx} (f(x)g(x)) &= f'(x)g(x) + f(x)g'(x) \\ \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ \frac{d}{dx} (f(g(x))) &= f'(g(x))g'(x). \end{aligned}$$

These become integration formulas:

$$\begin{aligned} \int (f'(x)g(x) + f(x)g'(x)) dx &= f(x)g(x) + c \\ \int \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} dx &= \frac{f(x)}{g(x)} + c \\ \int f'(g(x))g'(x) dx &= f(g(x)) + c. \end{aligned}$$

At first glance, these integration formulas look like they would be almost utterly useless for nearly every problem of antidifferentiation. Surprisingly (and maybe depressingly) they are instead our principal tools for integrating complicated functions. The first formula gives rise to a method called Integration by Parts, while the last is the basis of the even more important technique called Integration by Substitution, or Change of Variable. We won't look at these methods yet, but file the idea away, and remember you saw it here first.

### 5.1.3 Examples.

Antiderivatives of polynomials work just as they must.

$$\int (x^4 - 7x^3 + 2x - 9) dx = \frac{x^5}{5} - \frac{7}{4}x^4 + x^2 - 9x + c.$$

More complicated powers of  $x$  can be treated the same way.

$$\int \left( \frac{1}{x^2} + \sqrt{x} \right) dx = \int \left( x^{-2} + x^{1/2} \right) dx = \frac{x^{-1}}{-1} + \frac{x^{3/2}}{3/2} + c = -\frac{1}{x} + \frac{2}{3}x\sqrt{x} + c.$$

It's worth noting that there is exactly one power of  $x$  we cannot now integrate:  $x^{-1} = \frac{1}{x}$ . Our general formula would give

$$\int \frac{1}{x} dx = \int x^{-1} dx = \frac{x^0}{0} + c,$$

which can't possibly be right. We'll need to come back to this problem down the road. Amazingly, this one annoying exceptional case will turn out to lead us to some of the most important functions in mathematics.

Sometimes seemingly intractable problems can be rendered easy by some rewriting. Standard examples of this include things like

$$\begin{aligned} \int \left( \frac{1+x}{\sqrt{x}} \right) dx &= \int \left( \frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} \right) dx \\ &= \int \left( x^{-1/2} + x^{1/2} \right) dx \\ &= \frac{x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2} + c \\ &= 2\sqrt{x} + \frac{2}{3}x\sqrt{x} + c. \end{aligned}$$

Sometimes one can also play around and guess solutions. Here's an example where there are a couple of approaches. Suppose we want to compute

$$\int \sin x \cos x dx.$$

We might try to think of a function having  $\sin x \cos x$  as its derivative. A reasonable guess might be something like  $\sin^2 x$ . If we were to try this, we could compute its derivative:

$$\frac{d}{dx} \sin^2 x = 2 \sin x \cos x.$$

This isn't quite what we wanted, but it's close. Why not divide by 2? We'd then have

$$\frac{d}{dx} \left( \frac{\sin^2 x}{2} \right) = \frac{2 \sin x \cos x}{2} = \sin x \cos x,$$

so

$$\int \sin x \cos x dx = \frac{\sin^2 x}{2} + c.$$

Here's a second way to solve the same problem. We know that

$$\begin{aligned} \sin(2x) &= 2 \sin x \cos x \\ \frac{1}{2} \sin(2x) &= \sin x \cos x. \end{aligned}$$

Our problem is therefore to compute

$$\int \frac{1}{2} \sin(2x) dx.$$

We therefore need to think of a function whose derivative is  $\frac{1}{2} \sin(2x)$ . A reasonable first guess might be something like  $-\frac{1}{2} \cos(2x)$ , and indeed, computing the derivative of this gives

$$\frac{d}{dx} \left( -\frac{1}{2} \cos(2x) \right) = \frac{1}{2} \sin(2x) \cdot 2 = \sin(2x),$$

which is almost what we wanted. To get it exactly right, let's divide by 2. We then have

$$\frac{d}{dx} \left( -\frac{1}{4} \cos(2x) \right) = \frac{1}{4} \sin(2x) \cdot 2 = \frac{1}{2} \sin(2x),$$

so that

$$\int \sin x \cos x dx = \int \frac{1}{2} \sin(2x) dx = -\frac{1}{4} \cos(2x) + c.$$

Wait a minute, though. Didn't we just get 2 different answers for this integral? No, but it takes a bit of trigonometric cleverness to see that our two answers are equivalent.

$$\begin{aligned} -\frac{1}{4} \cos(2x) &= -\frac{1}{4} (\cos^2 x - \sin^2 x) \\ &= -\frac{1}{4} ((1 - \sin^2 x) - \sin^2 x) \\ &= \frac{1}{2} \sin^2 x - \frac{1}{4}, \end{aligned}$$

which has the form  $\frac{\sin^2 x}{2} + c$ . So our sense that there is order in the cosmos remains intact.

### 5.1.4 A First Differential Equation.

There turn out to be a vast number of situations in which one knows the derivative of a function, and one seeks the function itself. One standard example, one that calculus was invented to solve, is the dynamics of falling bodies. Let's start out remembering what we know about this problem.

If  $s(t)$  is the position of a moving object,  $v(t)$  is its velocity, and  $a(t)$  is its acceleration, then  $v(t) = s'(t)$  and  $a(t) = v'(t)$ . This means that

$$\begin{aligned}v(t) &= \int a(t) dt \\s(t) &= \int v(t) dt.\end{aligned}$$

If we can take antiderivatives, we can therefore compute the velocity and position from the acceleration.

This would be useful, since we also know, from the work of Galileo, that for a falling object near the earth, the acceleration is a constant,  $a(t) \doteq -9.8 \text{ m/s}^2$ . Let's use the approximation  $a(t) \doteq -10 \text{ m/s}^2$ , and try to do a problem that ties together nearly everything in this theory.

So here's the problem:

A person standing on top of a 40 meter tall building throws a ball up so that it goes up for one second before reaching its maximum height. It then falls past the starting position and eventually hits the ground. Questions one could now ask include

- (a) What's a formula for the velocity?
- (b) What's a formula for the position?
- (c) What's the maximum height?
- (d) When does the ball hit the ground?
- (e) How fast is it going when it hits the ground?
- (f) How will the police react to this?

And here are some solutions:

- (a) We know that  $a(t) = -10$ , and therefore that

$$v(t) = \int a(t) dt = \int -10 dt = -10t + c.$$

The problem is now to find a value for the constant  $c$ . We could do this if there were a time at which we knew the velocity. And there is: we know that at  $t = 1$ , the velocity is 0. Therefore

$$\begin{aligned}0 &= v(1) = -10(1) + c \\c &= 10.\end{aligned}$$

Thus,

$$v(t) = -10t + 10.$$

(b) We do the same thing to get the position from the velocity. We know

$$s(t) = \int (-10t + 10) dt = -5t^2 + 10t + k.$$

Again, the problem is to find the value of  $k$ . We can do this because we know that at  $t = 0$ ,  $s = 40$ . Thus,

$$40 = s(0) = -5(0^2) + 10(0) + k = k,$$

so that

$$s(t) = -5t^2 + 10t + 40.$$

(c) The maximum height occurs when  $t = 1$ , so it is  $s(1) = -5(1^2) + 10(1) + 40 = 45$  meters.

(d) The ball hits the ground when

$$\begin{aligned} -5t^2 + 10t + 40 &= 0 \\ t^2 - 2t - 8 &= 0 \\ (t - 4)(t + 2) &= 0 \\ t &= 4 \text{ or } t = -2. \end{aligned}$$

Obviously  $t = -2$  doesn't make physical sense, so the ball must hit the ground when  $t = 4$ .

(e) When  $t = 4$ , the velocity is  $v(4) = -10(4) + 10 = -30$  meters per second.

(f) Probably badly. It could also be some attorney's ticket to a major home improvement project.

## 5.2 Areas and Riemann Sums

### 5.2.1 Conceptual method.

If you think back on the beginnings of our work with derivatives, you can describe our method for computing the slopes of tangent lines as a process of 3 steps:

- (1) Give up. You can't compute slopes.
- (2) Try instead to approximate the slope by taking the slope of the secant line joining two close together points.
- (3) Now make the approximation into an exact result by taking the limit as the distance between the two points goes to 0.

We now want to start work on the second major problem of the calculus, the computation of the area under an arbitrary curve. Since we succeeded so well with the slope problem, it seems reasonable to try to use the same approach to compute the area under a curve. The three-fold method would then say,

- (1) Give up. You can't compute areas.
- (2) Try instead to approximate the area by breaking the region into a bunch of slices and approximating the area in each slice by the area of a rectangle

whose height is the height of the function at some point in the slice. Add up the areas of all these rectangles.

(3) Now make the approximation into an exact result by taking the limit as the number of slices approaches  $\infty$  and the width of each individual slice goes to 0.

Figure 1 shows a typical curve and the rectangles we might use in those slices.

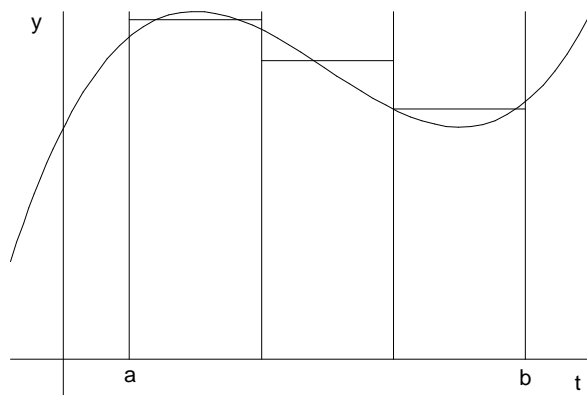


Figure 1: Approximating areas by slicing.

### 5.2.2 Formulas.

Suppose we want to try this method to compute the area under the graph of  $y = f(x)$  between  $x = a$  and  $x = b$ . We might begin by slicing the interval  $[a, b]$  into  $n$  slices of equal width  $\frac{b-a}{n}$ . The points dividing the slices would then be at

$$a + 0 \left( \frac{b-a}{n} \right), a + 1 \left( \frac{b-a}{n} \right), a + 2 \left( \frac{b-a}{n} \right), a + 3 \left( \frac{b-a}{n} \right), \dots, a + n \left( \frac{b-a}{n} \right).$$

Notice that the first of these points is  $a$ , and the last is  $b$ . If we build rectangles in each slice by using the height of the function  $f$  at the right hand end of the slice, then the first rectangle has width  $\frac{b-a}{n}$ , it has height

$$f \left( a + 1 \left( \frac{b-a}{n} \right) \right),$$

and therefore it has area

$$f \left( a + 1 \left( \frac{b-a}{n} \right) \right) \left( \frac{b-a}{n} \right).$$

The second rectangle has width  $\frac{b-a}{n}$ , height

$$f \left( a + 2 \left( \frac{b-a}{n} \right) \right),$$

and area

$$f\left(a + 2\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right),$$

and so on. The sum of the areas of all the rectangles is

$$\begin{aligned} A &= f\left(a + 1\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) + f\left(a + 2\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) + \dots \\ &\quad + f\left(a + n\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) \\ &= \left\{ f\left(a + 1\left(\frac{b-a}{n}\right)\right) + f\left(a + 2\left(\frac{b-a}{n}\right)\right) + \dots \right. \\ &\quad \left. + f\left(a + n\left(\frac{b-a}{n}\right)\right) \right\} \left(\frac{b-a}{n}\right). \end{aligned}$$

One could also write this in Sigma notation as

$$\begin{aligned} A &= \sum_{k=1}^n \left[ f\left(a + k\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) \right] \\ &= \left(\frac{b-a}{n}\right) \sum_{k=1}^n \left[ f\left(a + k\left(\frac{b-a}{n}\right)\right) \right]. \end{aligned}$$

The exact area will be the limit of this sum as  $n \rightarrow \infty$ .

A sketch of this situation is shown as Figure 2. In the Figure, we use the shorthand  $\Delta x$  to denote  $\frac{b-a}{n}$ .

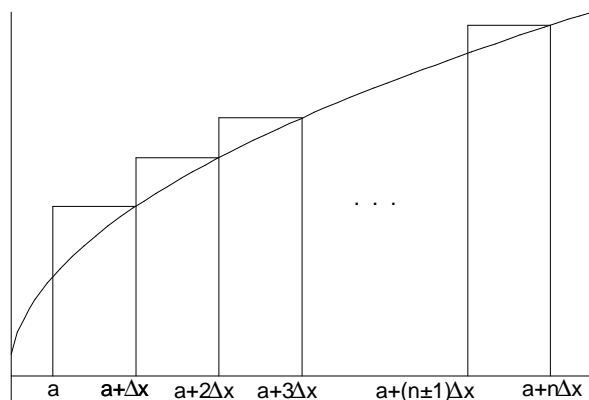


Figure 2: Right-hand rectangles.

### 5.2.3 Example 1: a straight line.

This approach seems to make sense, but it would certainly help to see some examples. Let's take the simplest case we could, the area under the straight

line  $y = 3x$  between  $x = 0$  and  $x = 1$ . We know the answer, since the area is just the area of a triangle; but let's make sure our general method works in this case. If we cut the interval  $[0, 1]$  into  $n$  equal width slices, then the cuts would be at the points

$$0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n} = 1.$$

The sum of the areas of the rectangles would therefore be

$$\begin{aligned} A &= f\left(\frac{1}{n}\right) \frac{1}{n} + f\left(\frac{2}{n}\right) \frac{1}{n} + f\left(\frac{3}{n}\right) \frac{1}{n} + \dots + f\left(\frac{n}{n}\right) \frac{1}{n} \\ &= \left\{ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right\} \frac{1}{n} \\ &= \left\{ \frac{3}{n} + \frac{6}{n} + \frac{9}{n} + \dots + \frac{3n}{n} \right\} \frac{1}{n} \\ &= \{1 + 2 + 3 + \dots + n\} \frac{3}{n^2}. \end{aligned}$$

Now if we want to work out this sum, we have to remember how to evaluate

$$S = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n.$$

There are lots of ways to do this. One trick is to write the sum in two different ways:

$$\begin{aligned} S &= 1 + 2 + 3 + \dots + (n-2) + (n-1) + n \\ S &= n + (n-1) + (n-2) + \dots + 3 + 2 + 1. \end{aligned}$$

If you add both these expressions in columns, you get

$$\begin{aligned} 2S &= (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) + (n+1) \\ &= n(n+1), \end{aligned}$$

from which it follows that

$$S = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n = \frac{n(n+1)}{2}.$$

Armed with this fact, we can now come back to the area we were trying to compute. We have

$$\begin{aligned} A &= \{1 + 2 + 3 + \dots + n\} \frac{3}{n^2} \\ &= \frac{n(n+1)}{2} \cdot \frac{3}{n^2} \\ &= \frac{3(n+1)}{2n} = \frac{3n+3}{2n}. \end{aligned}$$

This was our approximation to the area gotten by adding up the areas of a bunch of rectangles. The exact area should be the limit of this expression as

the number of slices approaches  $\infty$ , since in very narrow slices, the height of the rectangles and the height of  $y = f(x)$  are very nearly the same. In other words, the exact area should be

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{3n + 3}{2n} &= \lim_{n \rightarrow \infty} \frac{\frac{3n}{n} + \frac{3}{n}}{\frac{2n}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{3 + \frac{3}{n}}{2} = \frac{3}{2}. \end{aligned}$$

This is just what the formula  $A = \frac{1}{2}bh$  for the area of a triangle would have given us, so our method has delivered the right answer.

What we've just done is to help to substantiate our method for computing areas, by showing that it gives the correct answer in one of the few cases where we already know the area. In so doing, we've also computed the area of a triangle, not in the most complicated possible way known, but in one that comes pretty close.

#### 5.2.4 Example 2: a parabola.

Let's try a second example, one where we can't compute the area directly. Suppose we want the area under the parabola  $y = t^2$  between  $t = 0$  and  $t = x$ . That is, we want a generic right hand endpoint,  $x$ , for our interval.

If we cut the interval  $[0, x]$  into  $n$  equal width slices, then the cuts would be at the points

$$0, \frac{1x}{n}, \frac{2x}{n}, \frac{3x}{n}, \dots, \frac{nx}{n} = x.$$

The sum of the areas of the rectangles would therefore be

$$\begin{aligned} A &= f\left(\frac{1x}{n}\right) \frac{x}{n} + f\left(\frac{2x}{n}\right) \frac{x}{n} + f\left(\frac{3x}{n}\right) \frac{x}{n} + \dots + f\left(\frac{nx}{n}\right) \frac{x}{n} \\ &= \left\{ f\left(\frac{1x}{n}\right) + f\left(\frac{2x}{n}\right) + f\left(\frac{3x}{n}\right) + \dots + f\left(\frac{nx}{n}\right) \right\} \frac{x}{n} \\ &= \left\{ \frac{1^2x^2}{n^2} + \frac{2^2x^2}{n^2} + \frac{3^2x^2}{n^2} + \dots + \frac{n^2x^2}{n^2} \right\} \frac{x}{n} \\ &= \{1^2 + 2^2 + 3^2 + \dots + n^2\} \frac{x^3}{n^3}. \end{aligned}$$

Now if we want to work out this sum, we have to remember how to evaluate

$$S = 1^2 + 2^2 + 3^2 + \dots + n^2.$$

This one isn't as simple as the previous one. You probably never saw the formula in school. It turns out there is a formula, though, which is

$$S = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

It's not too hard to prove this by induction, but I'd rather not take the time to do so right now. Let's just assume the formula is right, and see where we end up.

The area we were trying to compute is

$$\begin{aligned} A &= \{1^2 + 2^2 + 3^2 + \dots + n^2\} \frac{x^3}{n^3} \\ &= \frac{n(n+1)(2n+1)}{6} \cdot \frac{x^3}{n^3} \\ &= \frac{2n^2 + 3n + 1}{6n^2} x^3. \end{aligned}$$

This was our approximation to the area gotten by adding up the areas of a bunch of rectangles. The exact area should be the limit of this expression as the number of slices approaches  $\infty$ , since in very narrow slices, the height of the rectangles and the height of  $y = f(x)$  are very nearly the same. In other words, the exact area should be

$$\lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} x^3 = \frac{2}{6} x^3 = \frac{x^3}{3}.$$

(Remember that if the numerator and denominator of a rational function have equal degrees, then the limit of the function at  $\infty$  is the ratio of the leading coefficients.)

Is this good or bad news? On the bright side, we got an area, the first really new area we've been able to compute. Even better, what we've done here is to start with the function  $f(t) = t^2$  and to compute the area function

$$F_0(x) = \frac{x^3}{3}.$$

Our conjecture from the lab was that  $F'_0(x) = f(x)$ , and that conjecture works perfectly here:

$$F'_0(x) = \frac{d}{dx} \left( \frac{x^3}{3} \right) = x^2 = f(x).$$

There is a bad side to this calculation, though: it wasn't very simple. In fact, we were only able to do it at all because of the formula I pulled out of the hat for

$$\sum_{k=1}^n k^2.$$

It doesn't look like  $f(x)$  would have to get much more complicated before we couldn't carry out the algebra at all. We therefore seem to have a theoretically successful method that may in practice be too complicated to use.

### 5.2.5 The Riemann Integral.

It doesn't look like we can often evaluate the sums that our techniques for computing areas produces. It also turns out that for technical reasons, using the right hand end of the intervals isn't always the best thing to do. With these thoughts in mind, Riemann used our idea to write a formal definition for the area under the graph of  $f(x)$  between  $x = a$  and  $x = b$ ,  $a < b$ .

Let  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ . Such a division of the interval  $[a, b]$  into  $n$  slices of possibly different lengths is called a partition of  $[a, b]$ . Denote the partition as  $P$ .

For every  $k$ ,  $1 \leq k \leq n$ , let

$$\Delta x_k = x_k - x_{k-1}$$

be the width of interval  $k$  in this partition. Also, for every  $k$ , let  $x_{k-1} \leq c_k \leq x_k$ , so that  $c_k$  is a point in slice  $k$  of the partition. An approximation to the area under  $f$  would then be

$$f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \dots + f(c_n)\Delta x_n = \sum_{k=1}^n f(c_k)\Delta x_k.$$

This sum is called a Riemann sum. Figure 3 shows a picture of one Riemann sum.

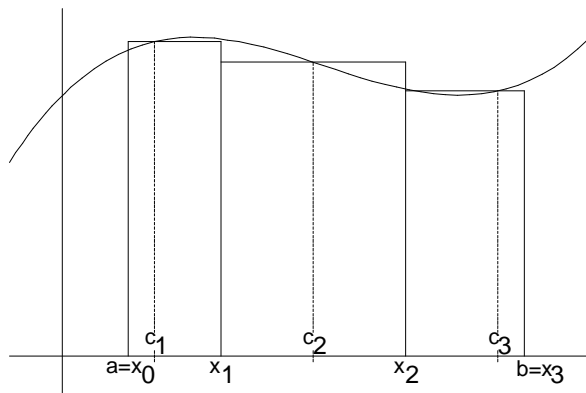


Figure 3: A typical Riemann sum.

A Riemann sum is an approximation to the exact area under the curve, gotten by adding up the areas of some rectangles. The exact area under the curve should be a limit of the Riemann sums as all the intervals in the partition  $P$  get small. One way to say this precisely is to let  $\|P\| = \max \{\Delta x_k : 1 \leq k \leq n\}$  be the largest of the slices in the partition  $P$ . Then the exact area under  $f$  should be

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k)\Delta x_k,$$

if this limit exists. This limit is called the Riemann integral of  $f$ , and is denoted

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k = \int_a^b f(x) dx.$$

Calculus books often do a few things with Riemann integrals in general, but we won't do much with them. It is perfectly reasonable to think of them as just slight variations on our earlier sums. After all, our earlier sums were just special cases of the sums above in which every  $\Delta x_k = \frac{b-a}{n}$  and in which every  $c_k = x_k$ . For simple continuous functions, there is nothing to be gained by working with general Riemann sums, though they are necessary if you want to find (or even, to define) the area under highly discontinuous functions.

### 5.2.6 A notational interlude.

The notation used for the Riemann integral looks odd, just as the notation for the antiderivative looked odd. If you think about it a bit, though, it becomes wonderful.

Leibniz' notation for the derivative was  $\frac{dy}{dx}$  because Leibniz thought of the derivative as really almost the slope  $\frac{\Delta y}{\Delta x}$  of a secant line. In Leibniz' notation, to move from the slope of the secant line to the slope of the tangent line, that is, to take a limit, you just change the Greek  $\Delta$  into a Latin  $d$ .

The same thing works with areas. An approximate area looks like

$$\sum_{k=1}^n f(c_k) \Delta x_k.$$

To get an exact area, you ought to turn the Greek letters into Latin letters. That is, the  $\Delta$  should become a  $d$ , and the  $\Sigma$  should become an  $S$ . The sum ran over points  $x_k = x$  and  $c_k = x$  between  $a$  and  $b$ . In the limit, we ought to be looking at a bunch of points spread over this interval. So we should write something like

$$S_a^b f(x) dx.$$

Stretch out the  $S$  to make it look more graceful, and you have

$$\int_a^b f(x) dx.$$

So for areas as for slopes, you denote the limit by just a change of alphabets. The integral sign is the residue of the sum, and the  $dx$  is the residue of the  $\Delta x$  after you take the limit.

Thinking of integrals roughly as infinite sums is not quite precise, but it is a great way to gain insights and intuitions about them.

### 5.2.7 Basic Formulas.

The two limits of sums we painfully computed above tell us that

$$\int_0^1 3x \, dx = \frac{3}{2},$$

and that

$$\int_0^x t^2 \, dt = \frac{x^3}{3}.$$

Getting more formulas of this sort seems to be a lot of work. Are there general formulas we can get by just thinking conceptually, though? Yes, here are a few:

$$\int_a^a f(x) \, dx = 0.$$

This just says that the area under  $y = f(x)$  between  $x = a$  and  $x = a$  is 0. This is certainly sensible, since the area is a rectangle of width 0.

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx.$$

This just says that if you take the area under  $f$  between  $x = a$  and  $x = b$ , and you add the area under  $f$  between  $x = b$  and  $x = c$ , you get the area under  $f$  between  $x = a$  and  $x = c$ , which surely makes sense.

In making this argument, we've assumed implicitly that  $a < b < c$ . What if this fails to be true? Well, our desire to make this formula always work would force us to say

$$\begin{aligned} \int_a^b f(x) \, dx + \int_b^a f(x) \, dx &= \int_a^a f(x) \, dx = 0 \\ \int_b^a f(x) \, dx &= -\int_a^b f(x) \, dx. \end{aligned}$$

This seems like an eccentric choice for a definition, but if we want our formulas so far to work without restriction, then we are forced to accept this definition. In mathematics, the definitions often follow the theorems, and are designed to make the theorems stay true; so it is not surprising that the community has settled on the agreement that area computed from right to left is the negative of area computed from left to right.

For any constant  $M$ ,

$$\int_a^b Mf(x) \, dx = M \int_a^b f(x) \, dx.$$

One could defend this geometrically by arguing that the area under  $y = Mf(x)$  must surely be  $M$  times the area beneath  $y = f(x)$ . After all, you'd approximate the area by adding up a bunch of rectangles, and multiplying the height of a

rectangle by  $M$  multiplies its area by  $M$  as well. One could also use Riemann's definition to say that

$$\begin{aligned} \int_a^b Mf(x) dx &= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n Mf(c_k) \Delta x_k \\ &= \lim_{\|P\| \rightarrow 0} M \sum_{k=1}^n f(c_k) \Delta x_k \\ &= M \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k \\ &= M \int_a^b f(x) dx. \end{aligned}$$

The first and last equalities here are the definition of the Riemann integral; the second is the distributive law; the third is the fact that constants can be pulled out of limits.

One that's more obvious algebraically than it is geometrically is

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

We could get this from the definition of the Riemann integral by writing

$$\begin{aligned} \int_a^b (f(x) + g(x)) dx &= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (f(c_k) + g(c_k)) \Delta x_k \\ &= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (f(c_k) \Delta x_k + g(c_k) \Delta x_k) \\ &= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k + \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n g(c_k) \Delta x_k \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx. \end{aligned}$$

To think about this geometrically takes a moment, though. Suppose  $f(x)$  and  $g(x)$  are the functions shown in Figures 4 and 5, resp.

Then the formula says that the area under  $y = f(x) + g(x)$ , shown in Figure 6, should be the sum of the areas of these 2 regions. Why should that be?

One way to start into this question is to look at the graphs of  $f(x)$  and  $f(x) + g(x)$  together, as in Figure 7.

We'd be done if we could understand why the top region in this Figure 7 has the same area as the area in Figure 5.

Algebraically, this is true because if we were to cut both regions into slices and approximate the areas with rectangles, in both cases the rectangle near  $x$  would have height about  $g(x)$ . If all the rectangles have the same size, then the total areas have to have the same size as well, and we're done.

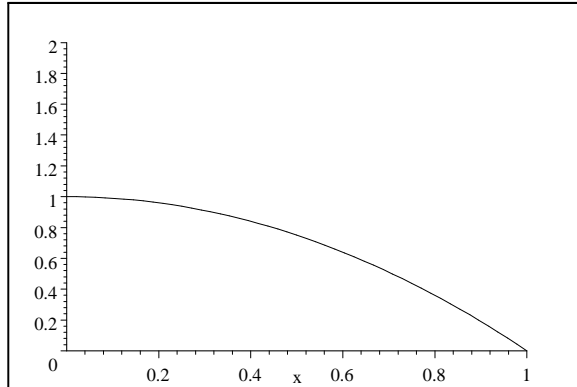


Figure 4:  $y = f(x)$ .

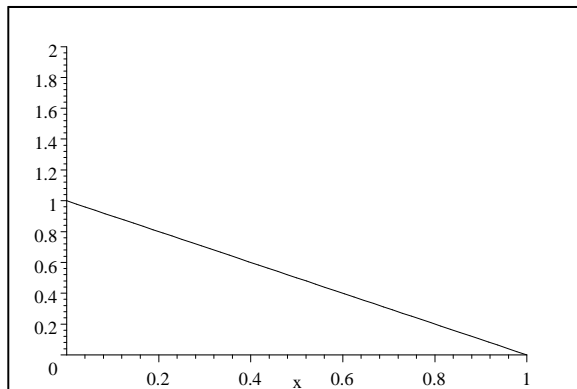


Figure 5:  $y = g(x)$ .

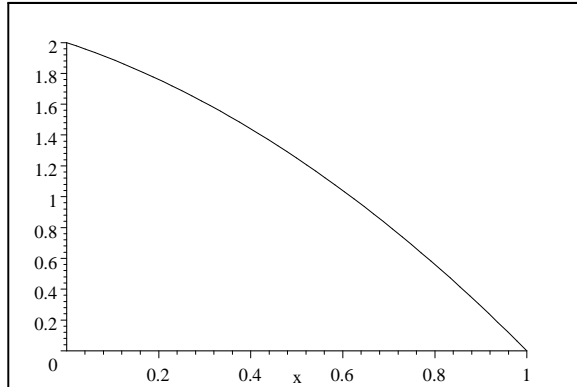


Figure 6:  $y = f(x) + g(x)$ .

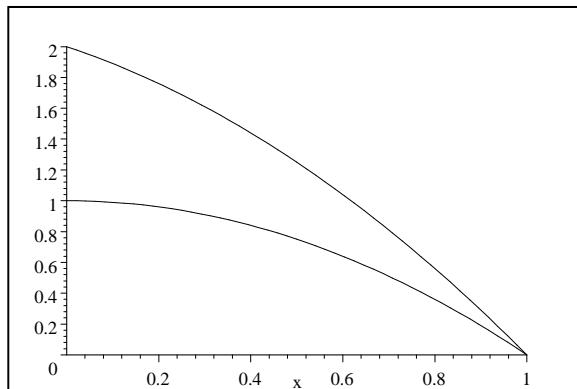


Figure 7:  $y = f(x)$  and  $y = f(x) + g(x)$ .

This observation is a special case of a general principle called Cavalieri's Principle, after an Italian mathematician of the generation before Newton. Cavalieri asserted that if you have two regions,  $A$  and  $B$ , and if every vertical line intersects  $A$  in a segment exactly as long as its intersection with  $B$ , then  $A$  and  $B$  have the same area. His segments play the same role as our rectangles. Cavalieri's argument for this was delightful: Imagine, he said, that each figure was drawn on cloth. Then for every thread, the length of that thread that meets  $A$  equals the length of that thread that meets  $B$ . Figures  $A$  and  $B$  must therefore have the same area, since they require the same amount of fabric to make.

### 5.3 The Fundamental Theorem of Calculus

Before I start on the Fundamental Theorem of Calculus, let me make a suggestion: In addition to these notes, it is very much worth reading your text or other treatments of the subject carefully. What we're doing now is the conceptual center of calculus. It's hard and subtle, and it's a good idea to look at it more than once, from different perspectives.

The Fundamental Theorem of Calculus is the theorem that tells us that the problem of finding slopes and the problem of finding areas are inverses of each other, thereby letting us finally make precise and prove a conjecture we've had since the first day of the class. The Fundamental Theorem lets us use antiderivatives to compute areas, and it unites the two different meanings we have for the integral.

#### 5.3.1 The Area function and its derivative.

In the lab, we looked at the function

$$F_a(x) = \int_a^x f(t) dt,$$

the area under the graph of  $y = f(t)$  between  $t = a$  and  $t = x$ . Our conjecture, for which we've been gathering stronger and stronger evidence, is that

$$F'_a(x) = \frac{d}{dx} \left( \int_a^x f(t) dt \right) = f(x).$$

This is one form (variously called the first form or the second form, depending on what course of logical development you're taking) of a result properly called the Fundamental Theorem of Calculus.

Calculus books often prove this theorem somewhat more precisely than I plan to, and from a slightly different perspective. Let me try to write precisely the argument for its truth that I hope you were led to discover in the lab.

To compute  $F'_a(x)$ , we need to find

$$\lim_{h \rightarrow 0} \frac{F_a(x+h) - F_a(x)}{h}.$$

The numerator of this difference quotient is the difference of two numbers.  $F_a(x+h)$  is the area under the graph of  $y = f(t)$  between  $t = a$  and  $t = x+h$ . Similarly,  $F_a(x)$  is the area under the graph of  $y = f(t)$  between  $t = a$  and  $t = x$ . The difference,  $F_a(x+h) - F_a(x)$  is therefore the area under the graph of  $y = f(t)$  between  $t = x$  and  $t = x+h$ . This is shown pictorially below in Figures 8 and 9.

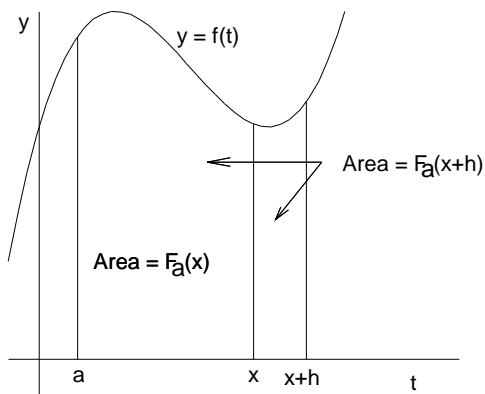


Figure 8:  $F_a(x)$  and  $F_a(x+h)$ .

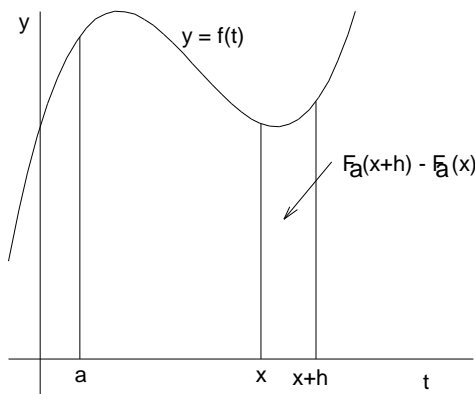


Figure 9:  $F_a(x+h) - F_a(x)$ .

When  $h$  is small, the difference  $F_a(x+h) - F_a(x)$  therefore represents an area which is very nearly a trapezoid whose width is  $h$  and whose heights are  $f(x)$  and  $f(x+h)$ . The trapezoid area formula therefore tells us that very nearly,

$$F_a(x+h) - F_a(x) = h \cdot \left( \frac{f(x) + f(x+h)}{2} \right).$$

This approximation should get better and better as  $h \rightarrow 0$ , since then the variation in height across the trapezoid should be less and less. As a result, it seems that we should have

$$\begin{aligned}
 F'_a(x) &= \lim_{h \rightarrow 0} \frac{F_a(x+h) - F_a(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[ \frac{1}{h} (F_a(x+h) - F_a(x)) \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{1}{h} \left( h \cdot \left( \frac{f(x) + f(x+h)}{2} \right) \right) \right] \\
 &= \lim_{h \rightarrow 0} \left( \frac{f(x) + f(x+h)}{2} \right) \\
 &= \frac{f(x) + f(x)}{2} = f(x).
 \end{aligned}$$

This proves the conjecture from the lab, and with it, this form of the Fundamental Theorem of Calculus.

There are a couple of points at which there has been a bit of friendly hand waving here. In computing the limit above, we've clearly assumed that  $f$  is a continuous function. We've also assumed rather vaguely that  $f$  doesn't change too wildly between  $x$  and  $x+h$ , so that the trapezoid is a reasonable approximation to the area. This can all be justified precisely using the Mean Value Theorem, but to me, it seems clearly sensible.

There is another form of the Fundamental Theorem of Calculus, (the second or first form, depending on whom you listen to), which says the following:

If  $F(x)$  is an antiderivative of  $f(x)$ , so that  $F'(x) = f(x)$ , then

$$\int_a^b f(t) dt = F(b) - F(a).$$

This result turns out to be a corollary of our earlier form of the Fundamental Theorem. We know that if

$$F_a(x) = \int_a^x f(t) dt,$$

then  $F'_a(x) = f(x)$ . Thus, both  $F_a(x)$  and  $F(x)$  are antiderivatives of  $f(x)$ . There is therefore some constant  $c$  such that

$$F_a(x) = F(x) + c.$$

All we need to do is find the value of  $c$ . To do this, we need to look for some value of  $x$  at which we know  $F_a(x)$ . The natural choice is  $x = a$ .  $F_a(a)$  is the area under the graph of  $f(t)$  between  $t = a$  and  $t = a$ . In other words,

$$F_a(a) = 0.$$

Plugging this into the earlier equation, we get

$$\begin{aligned}0 &= F_a(a) = F(a) + c \\c &= -F(a).\end{aligned}$$

Thus, for any  $x$ , we have

$$\int_a^x f(t) dt = F_a(x) = F(x) + c = F(x) - F(a).$$

Plugging in  $x = b$  gives the theorem.

### 5.3.2 A brief celebration.

This is a really cool result! Not only have we been wanting to prove this all semester, but it's very powerful. It means that we can now compute the area under the graph of any function whose antiderivative we can find. That's not quite every function, nor will it ever be; but it is quite a lot of functions.

### 5.3.3 Examples.

We had to work hard the other day to compute the area under the graph of  $y = t^2$  between  $t = 0$  and  $t = x$ . Now it's easy: An antiderivative of  $f(t) = t^2$  is  $F(t) = t^3/3$ , so

$$\int_0^x t^2 dt = F(x) - F(0) = \frac{x^3}{3} - \frac{0^3}{3} = \frac{x^3}{3}.$$

It's equally easy to find the area under the graph of  $y = t^n$  between  $t = 0$  and  $t = x$ . An antiderivative of  $f(t) = t^n$  is  $F(t) = \frac{t^{n+1}}{n+1}$ , so

$$\int_0^x t^n dt = F(x) - F(0) = \frac{x^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{x^{n+1}}{n+1}.$$

Two days ago, there were only two values of  $n$  (0 and 1) at which we knew the value of

$$\int_0^x t^n dt.$$

A moment ago, there were only 3 such values (0, 1, and 2). Now there are infinitely many, and it wasn't even hard. We're making fantastic progress.

Here's another example of our present prowess: What's the area under 1 hump of the function  $y = \sin x$ ? That is, what's

$$\int_0^\pi \sin x dx?$$

An antiderivative of  $f(x) = \sin x$  is  $F(x) = -\cos x$ , so

$$\int_0^\pi \sin x dx = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 2.$$

It doesn't get much simpler than that.

### 5.3.4 One bit of notation.

In order to cut down writing, people often write  $F(b) - F(a)$  as  $F(x)|_a^b$ , or as  $[F(x)]_a^b$ . This way, one can compute areas like this:

$$\begin{aligned}\int_1^3 (x^4 - 11x^3 + x^2 - 5) dx &= \left[ \frac{x^5}{5} - \frac{11x^4}{4} + \frac{x^3}{3} - 5x \right]_1^3 \\ &= \left[ \frac{3^5}{5} - \frac{11(3^4)}{4} + \frac{3^3}{3} - 5(3) \right] \\ &\quad - \left[ \frac{1^5}{5} - \frac{11(1^4)}{4} + \frac{1^3}{3} - 5(1) \right] \\ &= -\frac{3306}{20} - \left( -\frac{433}{60} \right) = -\frac{2494}{15}.\end{aligned}$$

### 5.3.5 FTC relates the 2 types of integral.

Let's try to think for a bit and reflect conceptually on the meaning of what we've now proven.

One way to understand the Fundamental Theorem of Calculus is to say that it ties together the two meanings we have for the integral sign. Let us remember what these are.

The definite integral,

$$\int_a^b f(t) dt,$$

is a number, denoting the area under the graph of  $f$  between  $t = a$  and  $t = b$ .

The indefinite integral,

$$\int f(x) dx,$$

is a function, the antiderivative of  $f$ .

It's reasonable and appropriate and in fact very insightful to ask why on earth we use almost the same symbolism for two such radically different notions? These two integrals aren't the same. They aren't even the same kind of thing: one is a number, and the other is a function.

The Fundamental Theorem of Calculus is the miraculous theorem that says that these two meanings of the integral sign are in fact intimately connected. It's worth thinking long enough about the meanings of the symbols that this connection becomes miraculous, even though that's hard because the notation, invented after the theorem was proved, makes it look not miraculous, but obvious.

The connection could be worked out in two ways. First, we could try to relate the two types of integral by somehow making the definite integral—a number, remember—into a function. Then it would at least be the same type of object as the indefinite integral. We do this by creating the area function,

$$F_a(x) = \int_a^x f(t) dt.$$

Our first form of the Fundamental Theorem says that

$$F'_a(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

In other words, it says that

$$\int_a^x f(t) dt$$

is an antiderivative of  $f(x)$ . Symbolically,

$$\int f(x) dx = \int_a^x f(t) dt + c.$$

So once we've turned the definite integral into a function, it turns out to be the same thing as the indefinite integral. At least, it's the same thing as close as it can be—it's one of the possible antiderivatives of  $f(x)$ .

The second way to connect the the two types of integral would be to try to turn the indefinite integral—a function—into a number, so that it would be the same type of object as the definite integral. The natural way to turn a function into a number is to take its value at a point. For instance, we turn the function  $\sin$  into a number by computing  $\sin \pi = 0$ . If we want to produce a number that might be the same as the number

$$\int_a^b f(x) dx,$$

though, we probably need to somehow combine the values of the function at the 2 points  $a$  and  $b$ . This is just what our second form of the Fundamental Theorem says: If

$$F(x) = \int f(x) dx,$$

then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Thus, turning the indefinite integral into a number by subtracting its values at  $b$  and at  $a$  makes it into exactly the same thing as the definite integral.

### 5.3.6 FTC says integrals and derivatives undo one another.

Yet another understanding of the Fundamental Theorem regards it as clarifying the senses in which differentiation and integration are inverse operations of one another. In one way, this is an obvious fact, since the indefinite integral is, after all, the antiderivative. But both forms of the Fundamental Theorem can also be interpreted as showing senses in which differentiation and definite integration invert one another.

In our first form of the Fundamental Theorem, we start with a function  $f(t)$ . We first take the definite integral to get

$$\int_a^x f(t) dt.$$

We then take the derivative to get

$$\frac{d}{dx} \int_a^x f(t) dt.$$

The Fundamental Theorem says that the result is  $f(x)$ , which is as close as possible to back to where we started—the original function with only the variable name changed. So the definite integral followed by the derivative in this way is essentially the identity operator on functions.

Our second form of the Fundamental Theorem is just a bit more subtle. Here, we start with a function  $F(x)$ . We first take the derivative to get  $f(x) = F'(x)$ . We then take a definite integral to get

$$\int_a^b F'(x) dx = F(b) - F(a).$$

This isn't quite the same thing we started with, but it's the closest we can come given that we have to end up with a number and not a function. So this form of the Fundamental Theorem says that as nearly as possible, the derivative followed by the definite integral is the identity operator on functions.

## 5.4 Integration by Substitution (Change of Variable)

Having seen how critical taking antiderivatives is, both to the problem of computing area and to problems in physics, we are led naturally to look for methods for computing the antiderivatives of more functions. So far, our methods for computing antiderivatives are limited. We know a few basic formulas, like

$$\begin{aligned} \int x^n dx &= \frac{x^{n+1}}{n+1} + c \\ \int \sin x dx &= -\cos x + c \\ \int \cos x dx &= \sin x + c \\ \int \sec^2 x dx &= \tan x + c \\ \int \sec x \tan x dx &= \sec x + c. \end{aligned}$$

We also know some general facts, like

$$\begin{aligned} \int (f(x) + g(x)) dx &= \int f(x) dx + \int g(x) dx \\ \int Mf(x) dx &= M \int f(x) dx. \end{aligned}$$

Armed with these facts, we can integrate any polynomial, and we can handle some things involving trig functions. We've also managed to integrate a few functions by just playing around. An example might be something like

$$\int \cos(2x + 3) dx.$$

We would guess that the integral might be something like  $\sin(2x + 3) + c$ . To test this guess, we take its derivative and see if we get  $\cos(2x + 3)$ . The chain rule tells us that

$$\frac{d}{dx} \sin(2x + 3) = \cos(2x + 3) \cdot 2.$$

This isn't quite what we were looking for, but it's close. If we divide both sides by 2, we get

$$\frac{d}{dx} \left( \frac{\sin(2x + 3)}{2} \right) = \frac{\cos(2x + 3) \cdot 2}{2} = \cos(2x + 3).$$

It follows that

$$\int \cos(2x + 3) dx = \frac{1}{2} \sin(2x + 3) + c.$$

There are plenty of functions, though, where these techniques fall short. How would we evaluate—or can one evaluate—integrals like

$$\begin{aligned} &\int \frac{x dx}{(x^2 + 1)^2} \\ &\int \sin^3 x \cos x dx \\ &\int x\sqrt{x+1} dx \end{aligned}$$

or even

$$\int \frac{dx}{x}?$$

It seems clear that it would be nice to have more methods of integration available.

#### 5.4.1 The bad news.

Having framed this desire, I have to tell you that there is some bad news involved with integration. Integration is harder than differentiation. We're not going to get rules like the product and quotient and chain rule that let us integrate any function whatever. There turn out to be simple integrals like

$$\int \sin(x^2) dx$$

that provably cannot be expressed in terms of elementary functions. There also are integrals that are startlingly complex, or startlingly unexpected:

$$\int \cos^5 x \, dx = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + c$$

$$\int \frac{dx}{1+x^2} = \arctan x + c.$$

If you're expecting integration to work out as simply as differentiation did, you're expecting too much. That being said, there is still a lot we can do to make integration more practical.

#### 5.4.2 Integration by Substitution.

The biggest technique for integration is called Substitution, or Change of Variable. We know that every differentiation formula is also an integration formula; integration by substitution turns out to be the integral form of the Chain Rule.

The Chain Rule says that

$$\frac{d}{dx}(f(u(x))) = f'(u(x))u'(x).$$

Turning this into an integration formula gives

$$\int f'(u(x))u'(x) \, dx = f(u(x)) + c.$$

This is the central formula for integration by substitution.

Now, it looks at first glance as though substitution might be almost completely useless. After all, how often would we expect to encounter an integral where the integrand just happened to have the form  $f'(u(x))u'(x)$ ? Even if an integrand did have this form, could we reliably recognize that fact? It's hard to be optimistic.

In fact, the situation isn't quite so bleak. For instance, look at the integral

$$\int \sin^3 x \cos x \, dx = \int (\sin x)^3 \cos x \, dx.$$

This integral would fit the form of our substitution formula if we took

$$\begin{aligned} u(x) &= \sin x \\ u'(x) &= \cos x \\ f'(x) &= x^3. \end{aligned}$$

To integrate it, we'd only need to find

$$f(x) = \int f'(x) \, dx = \int x^3 \, dx = \frac{x^4}{4} + c.$$

Then

$$\int \sin^3 x \cos x \, dx = f(u(x)) = \frac{u(x)^4}{4} + c = \frac{1}{4} \sin^4 x + c.$$

A second example of substitution might be something like this:

$$\int 3x^2 \sqrt{x^3 + 1} \, dx.$$

In this case, we could take

$$\begin{aligned} u(x) &= x^3 + 1 \\ u'(x) &= 3x^2 \\ f'(x) &= \sqrt{x}. \end{aligned}$$

In order to get the integral, we first compute

$$f(x) = \int f'(x) \, dx = \int \sqrt{x} \, dx = \frac{2}{3} x^{3/2} + c.$$

Given this, we can say

$$\int 3x^2 \sqrt{x^3 + 1} \, dx = f(u(x)) + c = \frac{2}{3} u(x)^{3/2} + c = \frac{2}{3} (x^3 + 1)^{3/2} + c.$$

Is it clear that we are gaining something by having substitution about?

### 5.4.3 Another View of Substitution.

There is another way to look at integration by substitution that is often easier to apply, though it takes a moment to see that it makes sense. We are starting out with the integral of a function of  $x$ .

$$\int f'(u(x)) u'(x) \, dx.$$

If by magic we could turn this into an integral of a function of a variable  $u$ ,

$$\int f'(u) \, du,$$

then we'd have an easy answer: the antiderivative of a derivative is the original function back again, so

$$\int f'(u) \, du = f(u) + c.$$

If we then remembered that  $u$  was really a function  $u(x)$ , we'd get the answer,

$$\int f'(u(x)) u'(x) \, dx = \int f'(u) \, du = f(u) + c = f(u(x)) + c.$$

This is just the formula for integration by substitution; so what substitution tell us is that this magic is all justified.

From this perspective, substitution is really a method for substituting  $u$  for  $u(x)$ , or for changing an integral with variable  $x$  into an integral with the variable  $u$ . This is where the name, “Change of Variable” comes from.

Now, here’s where the interesting formalism comes in. The original integral,

$$\int f'(u(x)) u'(x) dx,$$

contains both  $x$  and  $dx$ . In order to rewrite it as an integral in the variable  $u$ , we have to get rid of both the  $x$  and the  $dx$ , and to replace them with  $u$  and  $du$ . The first part of this is simple—we pick the function  $u(x)$  that we are going to use for  $u$ , and then we write  $u$  in place of this function wherever it appears. This would turn the integral into

$$\int f'(u) u'(x) dx.$$

We still have to get the  $dx$  out and a  $du$  in. We do this by the formal construction

$$\begin{aligned} \frac{du}{dx} &= u'(x) \\ du &= u'(x) dx. \end{aligned}$$

This seems to say it’s OK to replace  $u'(x) dx$  with  $du$ . If we do this, lo and behold, we get

$$\int f'(u) du,$$

which we can then rewrite as

$$f(u) + c = f(u(x)) + c,$$

just what we wanted.

#### 5.4.4 Is this legit?

It’s right to be suspicious of the somewhat crazy looking algebra above, where we move from  $\frac{du}{dx} = u'(x)$  to  $du = u'(x) dx$ . After all,  $\frac{du}{dx}$  isn’t a fraction; it’s just a notation for the derivative. How on earth can we justify taking it apart by multiplying by  $dx$ ? What kind of a thing is  $dx$ , anyway? Have mathematicians flipped?

This question is a deep and serious one that has been with calculus from its inception. There are a number of ways to answer it. On the simplest level, one could say that this crazy looking algebra is just a convenient mnemonic shorthand. We’ve justified Substitution by the use of the Chain Rule; so the validity of the method doesn’t depend on the meaningfulness of this algebra.

We're just using the algebra because we have shown that it gives the same result that we would get by Substitution, but that using this shorthand we can get the result more quickly and memorably. It's a convenience, not a theory.

This is the answer I was taught as a student, but it doesn't seem completely satisfactory. We'll use calculations like this so much that it seems depressing and vaguely dishonest just to shrug our shoulders and say that they are meaningless. Isn't there anything behind them? Leibniz and two centuries' worth of his successors would have answered that there certainly is something to this algebra. For Leibniz, remember,  $\frac{du}{dx}$  really was a fraction, the ratio of an infinitesimal  $\Delta u$  over an infinitesimal  $\Delta x$ . Since this thing that looks like a fraction really is a fraction, multiplying by the denominator is certainly legitimate; and all the algebra above makes sense. This perspective remains the most common conceptual standpoint of most non-mathematician users of the calculus even today.

Of course, there are real problems with Leibniz' answer. It's not at all clear what exactly these infinitesimals are. Indeed, if you think carefully about what the real numbers are, it becomes more and more clear that quantities functioning like Leibniz' infinitesimals cannot exist. This was already apparent at the time of the invention of the calculus. Bishop Berkeley famously described infinitesimals as "the ghosts of departed quantities," dead, that is, but not quite gone. He opposed calculus because of its lack of formal rigor, arguing that calculus led to atheism. One started out by believing crazy arguments in real analysis, Berkeley thought, and who knew where it would stop?

Now, I don't think calculus represents a threat to students' eternal salvation, or I would have found some other line of work; but I do think Berkeley's concerns about the logical problems of infinitesimals are completely legitimate. Indeed, calculus was a hugely successful set of techniques standing on deeply diseased foundations until the development of limits in the 19th Century. The new foundations for calculus in terms of limits, though, didn't change the results of the calculus, only their conceptual understanding. The new logical foundations left intact the old conclusions based on Leibniz' infinitesimals. So another answer to the question of the legitimacy of our crazy algebra is to say, calculations like this are OK because the careful logical foundations for calculus in terms of limits were carefully designed precisely to enable crazy arithmetic like this to work. Use it in good health!

Before I leave this topic and we start looking at examples, let me mention two more modern approaches to making sense of  $dx$  and  $du$ . One approach is due to Abraham Robinson in the late 1960's. Robinson was able using clever ideas in mathematical logic to construct models of the real numbers in which every theorem you know about the reals stays true, but in which there are actual infinitesimals (and their reciprocals, which are infinite numbers). This so-called Nonstandard Analysis meant that for the first time, one could use infinitesimals with complete rigor and without apology. It was a very unexpected and beautiful development.

There is also an approach to integration in which  $dx$  and  $du$  are regarded not as tiny numbers, but as a particular type of functions called differential forms.

I won't say more about how that works here, but you might meet these ideas in Multivariate Calculus.

#### 5.4.5 Examples.

Now that we've given some sort of defense for our crazy algebra, let's try it out on the integral

$$\int 3x^2 \sqrt{x^3 + 1} \, dx.$$

We'd say, we're going to do the change of variable

$$\begin{aligned} u &= x^3 + 1, \\ \frac{du}{dx} &= 3x^2 \\ du &= 3x^2 \, dx. \end{aligned}$$

Our goal is now to use these equations to rewrite the integral in terms of  $u$  and  $du$ . We do this by replacing  $\sqrt{x^3 + 1}$  by  $\sqrt{u}$ , and replacing  $3x^2 \, dx$  by  $du$ , to get the integral

$$\int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + c.$$

We then rewrite the result in terms of the old variable  $x$  to get

$$\frac{2}{3} (x^3 + 1)^{3/2} + c.$$

The fine thing about this approach is that any sort of algebra on the equations for  $u$  and  $du$  is legitimate, which often simplifies the task of doing the substitution. For instance, suppose we wanted to evaluate the integral

$$\int x \sin(x^2) \, dx.$$

We could try the substitution

$$\begin{aligned} u &= x^2 \\ du &= \frac{du}{dx} \, dx = 2x \, dx. \end{aligned}$$

This means that we can rewrite  $\sin(x^2)$  as  $\sin u$ , and that  $x \, dx = \frac{1}{2} du$ , so that the integral is

$$\int \frac{1}{2} \sin u \, du = -\frac{1}{2} \cos u + c = -\frac{1}{2} \cos(x^2) + c.$$

### 5.4.6 How do you pick the right $u$ ?

That's where the skill comes in. It's why integration is actually fun: you may have to try some different possibilities, and to be clever. There are a few rules of thumb one can use, though.

1. You ought to pick  $u(x)$  to be a function that appears inside another function, since that's the way it appears in  $f(u(x))$ .

2. You ought to pick  $u(x)$  to be a function whose derivative  $u'(x)$  (or something close to that) also appears somewhere in the integral. After all, we want to end up integrating  $f'(u(x))u'(x)$ .

3. One way to find  $u(x)$  is sometimes to look at the integral and say, "What's the worst thing about this integral?" You then let  $u(x)$  be that worst thing, and presto, it's gone!

4. Finally, you can just try anything for  $u(x)$  and see if it works. There aren't many possibilities in a typical integral, and it doesn't take long to try them all.

These principles still don't let you pick the right  $u$  in every single case. Would it occur to you, for instance, that the right choice of  $u$  in the integral

$$\int \sqrt{1-x^2} dx$$

is  $u = \arcsin x$ ? But in most cases, one or another of these rules of thumb will point you in the right direction.

### 5.4.7 More examples.

Given the integral

$$\int \frac{x+3}{(x^2+6x-5)^2} dx,$$

what would one do? A natural choice would be to let

$$\begin{aligned} u &= x^2 + 6x - 5 \\ du &= \frac{du}{dx} dx = (2x + 6) dx. \end{aligned}$$

This choice could be motivated by pretty much any of our rules of thumb. Once making it, we would say

$$\begin{aligned} \frac{1}{(x^2+6x-5)^2} &= \frac{1}{u^2} \\ (x+3) dx &= \frac{1}{2} du. \end{aligned}$$

The integral is therefore

$$\int \frac{x+3}{(x^2+6x-5)^2} dx = \frac{1}{2} \int \frac{du}{u^2} = -\frac{1}{2u} + c = -\frac{1}{2(x^2+6x-5)} + c.$$

Sometimes one has more than one choice of a substitution. Consider, for instance, our old friend

$$\int \cos x \sin x \, dx.$$

One could do this by saying, let

$$\begin{aligned} u &= \cos x \\ du &= \frac{du}{dx} dx = -\sin x \, dx. \end{aligned}$$

We can therefore rewrite  $\cos x$  as  $u$ , and  $\sin x \, dx$  as  $-du$ . The integral is thus

$$\int \cos x \sin x \, dx = -\int u \, du = -\frac{u^2}{2} + c = -\frac{1}{2} \cos^2 u + c.$$

An equally good choice would have been to let

$$\begin{aligned} u &= \sin x \\ du &= \frac{du}{dx} dx = \cos x \, dx. \end{aligned}$$

We can therefore rewrite  $\sin x$  as  $u$ , and  $\cos x \, dx$  as  $du$ . The integral is thus

$$\int \cos x \sin x \, dx = \int u \, du = \frac{u^2}{2} + c = \frac{1}{2} \sin^2 u + c.$$

Did we get the same answer? Yes, since

$$\frac{1}{2} \sin^2 x = \frac{1}{2} (1 - \cos^2 x) = -\frac{1}{2} \cos^2 x + \frac{1}{2} = -\frac{1}{2} \cos^2 x + c.$$

For some integrals, we need substitutions that seem too simple to be helpful. Here's an example where a very gentle touch is all that's required:

$$\int x\sqrt{x+1} \, dx.$$

The worst thing about this integral is the  $x+1$  under the square root. Except for that, we could just do it. So let's set

$$\begin{aligned} u &= x+1 \\ du &= \frac{du}{dx} dx = 1 \, dx = dx. \end{aligned}$$

Part of the integral is now easy to rewrite in the new variable. We replace  $\sqrt{x+1}$  with  $\sqrt{u}$ , and we replace  $dx$  with  $du$ . But what do we do with the  $x$ ? Well, we said before that algebra is always legitimate; so if  $u = x+1$ , then  $x = u-1$ . The integral is then

$$\int x\sqrt{x+1} \, dx = \int (u-1)\sqrt{u} \, du.$$

It takes a moment to realize that this is an improvement, but it is: we can just multiply out the new integral to get

$$\begin{aligned}\int (u\sqrt{u} - \sqrt{u}) du &= \int (u^{3/2} - u^{1/2}) du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + c \\ &= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c.\end{aligned}$$

This is hardly the first thing one would have written down in solving the integral. It's worth differentiating the answer, and making sure you really do get  $x\sqrt{x+1}$ .

#### 5.4.8 Using substitution repeatedly, or to simplify an integral.

It happens sometimes that one needs to use integration by substitution more than once to work out an integral. For instance, suppose you were confronted with the integral

$$\int (2x+5) \sin^4(x^2+5x-3) \cos(x^2+5x-3) dx.$$

A reasonable substitution would be

$$\begin{aligned}u &= x^2 + 5x - 3 \\ du &= \frac{du}{dx} dx = (2x+5) dx.\end{aligned}$$

After all,  $x^2 + 5x - 3$  is obnoxious, and it's inside the trig functions, and its derivative,  $2x + 5$ , is also present in the integral. When we rewrite the integral, we get

$$\int \sin^4 u \cos u du.$$

The new integral is way simpler than the old one, but it's still not so simple we can do it without another substitution. So what about letting

$$\begin{aligned}w &= \sin u \\ dw &= \frac{dw}{du} du = \cos u du.\end{aligned}$$

With this substitution, the integral becomes

$$\begin{aligned}\int w^4 dw &= \frac{w^5}{5} + c \\ &= \frac{\sin^5 u}{5} + c \\ &= \frac{\sin^5(x^2+5x-3)}{5} + c,\end{aligned}$$

where we've worked our way backwards through the changes of variable to arrive at a function of  $x$ .

Did we really need 2 substitutions here? No. We could have used a single substitution,

$$\begin{aligned} u &= \sin(x^2 + 5x - 3) \\ du &= \frac{du}{dx} dx = \cos(x^2 + 5x - 3) \cdot (2x + 5) dx. \end{aligned}$$

This would have done the integral in one step, had one been insightful enough to see it from the beginning.

It also sometimes happens that we end up dealing with an integral we can't completely work out, but that a substitution can simplify. For instance, suppose we needed to evaluate

$$\int \frac{\sec^2 x dx}{1 + \tan^2 x}.$$

A reasonable start would be to try the substitution

$$\begin{aligned} u &= \tan x \\ du &= \frac{du}{dx} dx = \sec^2 x dx. \end{aligned}$$

The integral then becomes

$$\int \frac{du}{1 + u^2}.$$

At this point, we're stuck for right now. We don't know how to do the integral we're left with. But if we have to approximate the answer numerically, wouldn't you rather work with the integral in this form than in the other form?

Incidentally, there is a better way to do the integral. Remember that

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\ \tan^2 x + 1 &= \sec^2 x. \end{aligned}$$

The integral is thus

$$\int \frac{\sec^2 x dx}{1 + \tan^2 x} = \int \frac{\sec^2 x dx}{\sec^2 x} = \int dx = x + c.$$

This actually leads us to an interesting and surprising observation. We've just seen that if  $u = \tan x$ , then

$$\int \frac{du}{1 + u^2} = x + c = \arctan u + c,$$

which is equivalent to the derivative formula

$$\frac{d}{du} \arctan u = \frac{1}{1 + u^2}.$$

Can you prove this formula without using integration by substitution?

#### 5.4.9 Integrals needing some algebra first.

Let's look again at the integral

$$\int \cos^5 x \, dx,$$

for which I pulled a value out by magic at the beginning of this section. I could I have done this? There doesn't seem to be an obvious substitution for this integral, but here's a lovely trick: You know that  $\cos^2 x = 1 - \sin^2 x$ , which implies that  $\cos^4 x = (1 - \sin^2 x)^2$ . The integral is thus

$$\int (1 - \sin^2 x)^2 \cos x \, dx.$$

To do this, wouldn't one naturally take

$$\begin{aligned} u &= \sin x \\ du &= \frac{du}{dx} dx = \cos x \, dx. \end{aligned}$$

The integral would then become

$$\begin{aligned} \int (1 - u^2)^2 du &= \int (1 - 2u^2 + u^4) du \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c \\ &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c, \end{aligned}$$

the same mysterious answer mentioned near the beginning of these notes.

This same technique works for every odd power of  $\sin x$  or of  $\cos x$ .

#### 5.4.10 Substitution with definite integrals.

So far, all our integrals have been indefinite integrals. What if we were to try doing substitution on a definite integral? You'd be tempted just to do something wrong like this: The substitution

$$\begin{aligned} u &= x^2 + 5x + 2 \\ du &= \frac{du}{dx} dx = (2x + 5) dx \end{aligned}$$

lets us write

$$\int_0^1 (x^2 + 5x + 2)^3 (2x + 5) dx = \int_0^1 u^3 du = \left[ \frac{u^4}{4} \right]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}.$$

This can't possibly be right, though, since the minimum value of the function  $(x^2 + 5x + 2)^3 (2x + 5)$  on the interval  $0 \leq x \leq 1$  is 40, so that the integral over an interval of length 1 has to be at least 40.

What went wrong?

One way to see why this calculation failed is to work a bit more carefully. If we were doing the indefinite integral, we could indeed write

$$\begin{aligned}\int (x^2 + 5x + 2)^3 (2x + 5) dx &= \int u^3 du \\ &= \frac{u^4}{4} + c \\ &= \frac{(x^2 + 5x + 2)^4}{4} + c.\end{aligned}$$

The Fundamental Theorem of Calculus would now say that since this is an antiderivative of the integrand, we have

$$\begin{aligned}\int_0^1 (x^2 + 5x + 2)^3 (2x + 5) dx &= \left[ \frac{(x^2 + 5x + 2)^4}{4} \right]_0^1 \\ &= \frac{(1^2 + 5(1) + 2)^4}{4} - \frac{(0^2 + 5(0) + 2)^4}{4} \\ &= \frac{8^4}{4} - \frac{2^4}{4} = 1020.\end{aligned}$$

We can now see the problem with our earlier calculation. We plugged in  $u = 0$  and  $u = 1$ , when we should have plugged in  $x = 0$  and  $x = 1$ .

One way to do definite integrals with substitution is therefore to find the indefinite integral, write it back in terms of the original variable  $x$ , and then plug in the endpoint values for  $x$  and take the difference.

Here's an equivalent but simpler procedure. Imagine we have the initial integral written out more explicitly than usual, like this:

$$\int_{x=0}^{x=1} (x^2 + 5x + 2)^3 (2x + 5) dx.$$

The substitution

$$\begin{aligned}u &= x^2 + 5x + 2 \\ du &= \frac{du}{dx} dx = (2x + 5) dx\end{aligned}$$

would let us rewrite this as

$$\int_{x=0}^{x=1} (x^2 + 5x + 2)^3 (2x + 5) dx = \int_{x=0}^{x=1} u^3 du.$$

Ah, here's the problem with our first calculation: We ought to change the endpoints of integration to reflect the new variable. How? Well, when  $x = 0$ , we have  $u = 0^2 + 5(0) + 2 = 2$ , and when  $x = 1$ , we have  $u = 1^2 + 5(1) + 2 = 8$ . We should therefore rewrite the integral as

$$\int_{x=0}^{x=1} u^3 du = \int_{u=2}^{u=8} u^3 du = \int_2^8 u^3 du = \left[ \frac{u^4}{4} \right]_2^8 = \frac{8^4}{4} - \frac{2^4}{4} = 1020.$$

Stare at this a bit until you see why it is that the method of running the indefinite integral back to  $x$  and then plugging in the old endpoints for  $x$  results in exactly the same result as leaving the integral in terms of  $u$  and plugging in new endpoints for  $u$ .

Here's another example of this sort of calculation, written the way one would normally see it set down. To compute

$$\int_0^{\pi/2} (\cos x) \sqrt{\sin x} \, dx,$$

we ought to use the substitution

$$\begin{aligned} u &= \sin x \\ du &= \frac{du}{dx} dx = \cos x \, dx. \end{aligned}$$

The integral is thus

$$\int_0^{\pi/2} (\cos x) \sqrt{\sin x} \, dx = \int_0^1 \sqrt{u} \, du = \left[ \frac{2}{3} u^{3/2} \right]_0^1 = \frac{2}{3} (1^{3/2}) - \frac{2}{3} (0^{3/2}) = \frac{2}{3}.$$

The endpoints changed when we did the change of variable because when  $x = 0$ , we have  $u = \sin x = \sin 0 = 0$ , and when  $x = \pi/2$ , we have  $u = \sin x = \sin \frac{\pi}{2} = 1$ .

#### 5.4.11 Do we need this?

Students often don't like this change of variable in definite integrals. It's confusing at first, and it seems to many people that it's easier just to work out the indefinite integral in terms of  $x$ , and then to plug in values. Let me end this section with an example showing that you can't always do this. Sometimes you really need to change the limits of integration in the definite integral.

Here's the example: look at

$$\int_{-1}^1 2x \sin \left( (x^2 - 1)^2 \right) \, dx.$$

The natural thing to do to this integral would be to let

$$\begin{aligned} u &= x^2 - 1 \\ du &= \frac{du}{dx} dx = 2x \, dx. \end{aligned}$$

The indefinite integral would then become

$$\int 2x \sin \left( (x^2 - 1)^2 \right) \, dx = \int \sin (u^2) \, du.$$

The trouble now is that we can't do the new integral. In fact, there is provably no indefinite integral for  $\sin (u^2)$  that can be written in terms of elementary functions (polynomials, trig functions, logs and exponentials). You're completely stuck.

Is the definite integral impossible without resorting to numerical methods, then? Not at all. When  $x = -1$ ,  $u = (-1)^2 - 1 = 0$ ; and when  $x = 1$ ,  $u = 1^2 - 1 = 0$ . The definite integral is therefore

$$\int_{-1}^1 2x \sin((x^2 - 1)^2) dx = \int_0^0 \sin(u^2) du = 0,$$

since the area of a line segment is 0. Conclusion: you do need to come to terms with change of variable in definite integrals. Sometimes it's the only game in town.