6 Logarithms and Exponential Functions

6.1 Introduction and Review.

6.1.1 Laws of Exponents.

First of all, let's remember the basic algebraic properties of algebraic and logarithmic functions.

All the definitions and theorems about exponents come from the fundamental fact

$$a^n a^m = a^{n+m}.$$

This principle is obvious if n and m are positive integers, as is the related principle that

$$(a^n)^m = a^{nm}.$$

Everything else about exponential functions grows from a desire to make these principles hold regardless of whether or not n and m are positive, or integral, or rational. For instance, taking $n = 0$ would give

$$a^0 a^n = a^{0+n} = a^n,$$

which means that if we want the exponent of the product to be the sum of the exponents, we have to define $a^0 = 1$. Similarly, we want to have

$$a^n a^{-n} = a^0,$$

which, given that we now know $a^0 = 1$, means that we have to define

$$a^{-n} = \frac{1}{a^n}.$$

Similarly, a desire to make $(a^n)^m = a^{nm}$ hold universally means that we must have

$$(a^{1/q})^q = a^{q/q} = a^1 = a,$$

which means that we have to define

$$a^{1/q} = \sqrt[q]{a},$$

which means, in turn, that

$$\sqrt[q]{a^p} = (a^p)^{1/q} = a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p.$$

All these definitions may have been presented to you in the past just as facts to learn—and you should just learn them—but isn't it interesting to see how they arise. Like so many things in mathematics, the theorems came first—we knew what properties of integer powers we wanted to generalize—and the definitions were designed to guarantee the theorems we wanted.

A truly observant student will notice that nothing we have done tells us anything about the value of a^r if r is irrational. You've probably believed all your life that expressions like $2^{\sqrt{2}}$ or 10^{π} make sense, but have you ever actually seen them defined? Could you propose definitions now? Think about this question; we'll need to come back to it. Part of the reason we're working with logs and exponential functions is to be able to answer it, which we'll do in a rather unexpected way. Stay tuned.

6.2 Logarithms.

Logarithms are just exponential functions done backwards. They have the same relation to exponential functions that square roots have to squares. That is,just as

$$y = \sqrt[n]{x} \text{ if and only if } x = y^n,$$

we also have

$$y = \log_a x \text{ if and only if } x = a^y.$$

The number $\log_a x$ is the power to which you raise a in order to get x. Think this through, and you'll have the two identities

$$\begin{aligned} a^{\log_a x} &= x, \\ \log_a(a^x) &= x. \end{aligned}$$

From the definition of logarithms, we get immediately the sort of facts one uses to build intuitions about logs:

$$\begin{aligned} \log_a 1 &= 0 \\ \log_a a &= 1 \\ \log_a(a^2) &= 2 \\ \log_a(a^3) &= 3. \end{aligned}$$

Every law of exponents is equivalent to a law of logarithms. The most fundamental of these comes from the observation that

$$a^{\log_a(xy)} = xy = (a^{\log_a x})(a^{\log_a y}) = a^{\log_a x + \log_a y}.$$

This means that we must have

$$\log_a(xy) = \log_a x + \log_a y.$$

The log of a product is the sum of the logs of the factors.

Similarly, we have

$$a^{\log_a(x^n)} = x^n = (a^{\log_a x})^n = a^{n \log_a x},$$

which means that we must have

$$\log_a(x^n) = n \log_a x.$$

From these identities, we can derive others of use. For instance,

$$\log_a x + \log_a \frac{1}{x} = \log_a \frac{x}{x} = \log_a 1 = 0;$$

so

$$\log_a \frac{1}{x} = -\log_a x.$$

See if you can also derive the law

$$\log_a \frac{x}{y} = \log_a x - \log_a y.$$

There is a final identity that is essential in computing logs to eccentric bases:

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

This lets us compute logs to the base b as long as there is some base a for which we can compute logs. To see that this identity is true, we observe that

$$a^{\log_a x} = x = b^{\log_b x} = (a^{\log_a b})^{\log_b x} = a^{\log_a b \log_b x}.$$

From this, we infer that

$$\log_a x = \log_a b \log_b x.$$

Divide both sides by $\log_a b$, and you're set.

It's worth looking quickly at graphs of exponential and logarithmic functions. Figure 1 shows together the graphs of $y = 2^x$ and $y = \log_2 x$.

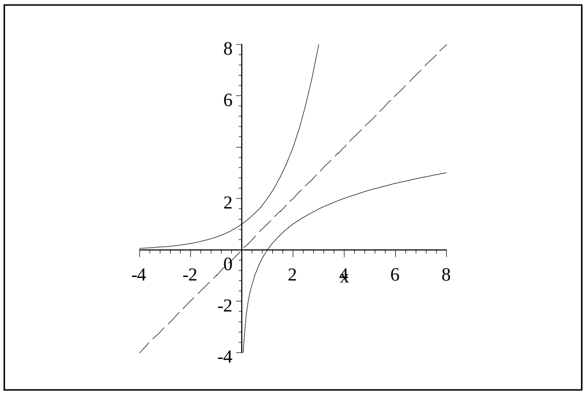

Figure 1: $y = 2^x$ and $y = \log_2 x$.

Since $y = \log_2 x$ and $x = 2^y$ mean the same thing, the graph of $y = \log_2 x$ and the graph of $y = 2^x$ should be related by interchanging the x and y axes. Geometrically, this interchange is achieved by reflecting the plot across the line $y = x$, which is shown dashed in Figure 1.

Logs and exponential functions to other bases look similar to the graphs in Figure 1. Figure 2 shows the graphs of $y = 2^x$ and $y = 8^x$ on the same set of axes. 8^x grows much faster for positive x and shrinks much faster for negative x. The corresponding logarithms are shown in Figure 3.

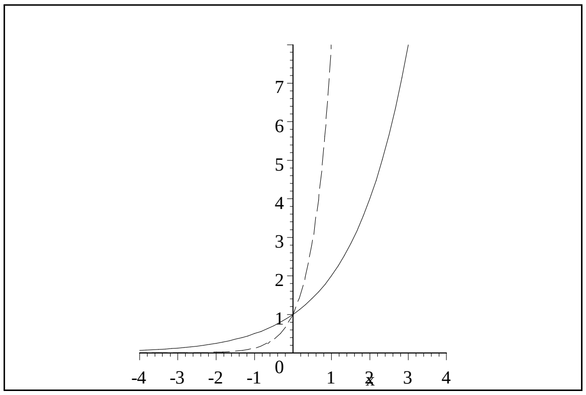

Figure 2: $y = 2^x$ and $y = 8^x$.

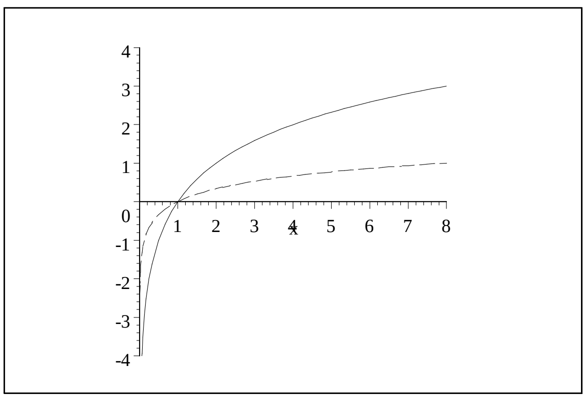

Figure 3: $y = \log_2 x$ and $y = \log_8 x$.

6.3 Why are we doing this?

What has motivated us to think now about logs and exponential functions? Well, it turns out that we haven't quite done calculus as completely as we've been claiming, and that these functions are a big gap in our knowledge. We've been claiming for a long time, for instance, that we can differentiate any function we can write down. OK, if we're so cool, what's

$$\frac{d}{dx}(2^x)?$$

One might guess that the answer would be $x2^{x-1}$, though one ought to be at least a bit uneasy about this answer. After all, 2^x, where the variable is in the exponent and the base is a constant, is really different from x^2, where the variable is the base and the exponent is a constant. In any case, this answer is hopelessly wrong, since it would predict that if $f(x) = 2^x$, then $f'(0) = 0$, and that $f'(x) < 0$ if $x < 0$. Both these claims are obviously wrong, as a glance at the graph of 2^x shows at once.

So we really were premature in declaring victory in differentiation. We can't differentiate 2^x, and we can't differentiate $\log_2 x$. There's still work to be done.

6.4 The Natural Log

The solution to the problem of differentiating exponential and logarithmic functions comes from an unexpected source—our only remaining integration problem involving powers of x. Remember that as long as $n \neq -1$, we have

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c.$$

When $n = -1$, this formula makes no sense; and so far, we have not had any means for evaluating the integral

$$\int \frac{dx}{x}.$$

On the other hand, we know that this antiderivative must exist, since by the Fundamental Theorem of Calculus, one antiderivative of $\frac{1}{x}$ is the area function

$$\int_1^x \frac{dt}{t}.$$

Our goal at the moment is to see what we can learn about this area function.

If you plot the area under the graph of $\frac{1}{t}$ between $t = 1$ and $t = x$, you get the graph shown in Figure 4, which looks a lot like a logarithm.

The amazing thing is that this area function is a logarithm! I find this one of the most startling results in elementary calculus. All the antiderivatives of powers of x are just simple and uninteresting multiples of other powers of x. Then, completely out of the blue, there is one exception: the antiderivative of x^{-1}, alone out of all the infinitely many powers of x, turns out to be a completely different kind of function, a logarithm. This one integral lives in an utterly different universe from all the others, and opens up whole new mathematical realms. We've rather casually ignored this one exceptional integral up until now, but one ignores such oddballs at one's peril. Listening to the message of

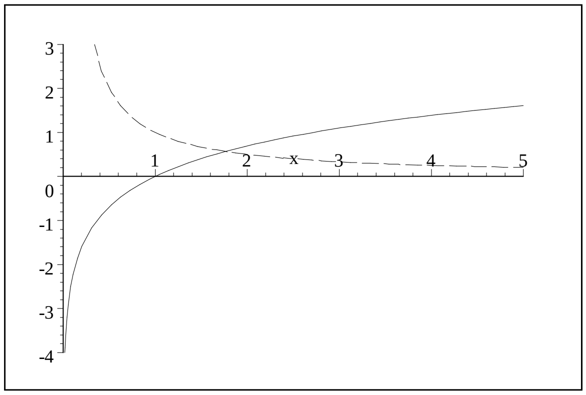

Figure 4: $\frac{1}{t}$ and $\int_1^x \frac{dt}{t}$.

this one function out of infinitely many that marches to the beat of a different drummer opens new worlds to our view.

OK, so let's define the natural log function as

$$\ln x = \int_1^x \frac{dt}{t},$$

the area under the graph of $y = \frac{1}{t}$ between $t = 1$ and $t = x$. The graph of this function is at least roughly like the graph of $y = \log_a x$ for some a. We now need to show that in fact, $\ln x$ is a logarithm function.

Let's start out by showing that $\ln x$ has the right algebraic properties to be a logarithm to some base. One of these is already obvious:

$$\ln 1 = \int_1^1 \frac{dt}{t} = 0.$$

More substantively, though, we'd like to show things like $\ln(ab) = \ln a + \ln b$. To do this, we can start out by noticing that

$$\ln(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} = \ln a + \int_a^{ab} \frac{dt}{t}.$$

We'd therefore be done if we could show that

$$\int_a^{ab} \frac{dt}{t} = \ln b.$$

This turns out, though, just to be an easy integration by substitution: the substitution

$$\begin{aligned} u &= \frac{t}{a} \\ du &= \frac{dt}{a}, \end{aligned}$$

or, to put it more conveniently,

$$\begin{aligned} t &= au \\ dt &= a\,du \end{aligned}$$

turns the integral into

$$\int_a^{ab} \frac{dt}{t} = \int_1^b \frac{a\,du}{au} = \int_1^b \frac{du}{u} = \ln b,$$

so

$$\ln(ab) = \ln a + \int_a^{ab} \frac{dt}{t} = \ln a + \ln b.$$

Similarly, we'd like to know that $\ln \frac{1}{a} = -\ln a$. This again is a consequence of substitution. The substitution

$$\begin{aligned} u &= at \\ du &= a\,dt, \end{aligned}$$

or, to put it more conveniently,

$$\begin{aligned} t &= \frac{u}{a} \\ dt &= \frac{du}{a} \end{aligned}$$

lets us write

$$\ln \frac{1}{a} = \int_1^{1/a} \frac{dt}{t} = -\int_{1/a}^1 \frac{dt}{t} = -\int_1^a \frac{du}{a(u/a)} = -\int_1^a \frac{du}{u} = -\ln a.$$

The identity $\ln(a^n) = n \ln a$ is also a consequence of a substitution like this. If we use the substitution

$$\begin{aligned} u &= t^{1/n} \\ du &= \frac{t^{\frac{1}{n}-1}}{n}\,dt, \end{aligned}$$

or, more simply,

$$\begin{aligned} t &= u^n \\ dt &= nu^{n-1}\,du, \end{aligned}$$

then we find

$$\ln(a^n) = \int_1^{a^n} \frac{dt}{t} = \int_1^n \frac{nu^{n-1}\,du}{u^n} = n \int_1^a \frac{du}{u} = n \ln a.$$

All this makes it seem highly likely that the function $\ln x$ really is a log to some base. But what is the base of the natural logarithms? Well, if $\ln x$ is really

$\log_e x$ for some number e, then it must be that $\ln e = \log_e e = 1$. In other words, e must be the number for which

$$1 = \ln e = \int_1^e \frac{dt}{t}.$$

To approximate e, we could begin by just using rectangles to approximate the integral. For instance, if we use left rectangles of width $\frac{1}{4}$, we find that

$$\int_1^{2.25} \frac{dt}{t} < \frac{1}{4}\left\{\frac{1}{1} + \frac{1}{5/4} + \frac{1}{3/2} + \frac{1}{7/4} + \frac{1}{2} + \frac{1}{9/4}\right\} < 1,$$

so that $e > 2.25$; and if we use right rectangles of width $\frac{1}{4}$, we find that

$$\int_1^3 \frac{dt}{t} > \frac{1}{4}\left\{\frac{1}{5/4} + \frac{1}{3/2} + \frac{1}{7/4} + \frac{1}{2} + \frac{1}{9/4} + \frac{1}{5/2} + \frac{1}{11/4} + \frac{1}{3}\right\} > 1$$

so that $e < 3$. Better calculations with more rectangles or with other expressions for e give

$$e \doteq 2.718281828459.$$

If $\frac{p}{q}$ is any rational number, then the identities we've just proved for the natural log function give

$$\ln(e^{p/q}) = \left(\frac{p}{q}\right)\ln e = \left(\frac{p}{q}\right)1 = \frac{p}{q}.$$

Thus, the natural log function seems to be the log to the base e, for every number $x = \frac{p}{q}$ at which e^x is defined.

6.4.1 $\ln(x)$ and integrals.

We now have a function $\ln x$ defined for $x > 0$ and having the property that when $x > 0$, $\frac{d}{dx}\ln x = \frac{1}{x}$, i.e., that when $x > 0$,

$$\int \frac{1}{x}\,dx = \ln x + c.$$

What can we say about the integral of $\frac{1}{x}$ if $x < 0$? Well, if $x < 0$, then $-x > 0$; so by the Chain Rule,

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

This means that for negative x,

$$\int \frac{1}{x}\,dx = \ln(-x) + c.$$

We can roll these two observations together to say that for any $x \neq 0$,

$$\int \frac{1}{x}\,dx = \ln|x| + c.$$

Thus, our definite integral for the function $\frac{1}{x}$ starting at $x = 1$ also yields an indefinite integral for $\frac{1}{x}$ valid at every $x \neq 0$.

Armed with this, we can do lots of other integrals as well. For instance, consider

$$\int \frac{2x-3}{x^2-3x+1}\,dx.$$

This integral can be reduced by the substitution

$$\begin{aligned} u &= x^2 - 3x + 1 \\ du &= (2x-3)\,dx \end{aligned}$$

to give

$$\int \frac{du}{u} = \ln|u| + c = \ln\left|x^2 - 3x + 1\right| + c.$$

Another clever example of the same technique lets us integrate

$$\int \tan x\,dx.$$

We rewrite this integral as

$$\int \frac{\sin x}{\cos x}\,dx,$$

then use the substitution

$$\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \end{aligned}$$

to write this as

$$-\int \frac{du}{u} = -\ln|u| + c = -\ln|\cos x| + c.$$

The first thing you would have guessed, right?

6.4.2 Using $\ln x$ to define exponentiation.

When we talked informally about exponential and logarithmic functions, we described them as inverses of one another, just as the square root and square functions are inverses of one another. If the functions e^x and $\ln x$ were to be inverses of one another, what would we need? Fundamentally, all we would require algebraically would be that these two functions undo one another, that is, that

$$\ln(e^x) = x$$

and that

$$e^{\ln x} = x.$$

Do we know these facts? Well, we know that if $x = \frac{p}{q}$ is rational, then

$$\ln(e^x) = \ln(e^{p/q}) = \frac{p}{q}\ln(e) = \frac{p}{q}1 = \frac{p}{q} = x.$$

We don't know that this same formula holds if x is some irrational number, like $\sqrt{2}$ or like π, but the reason we don't know it is somewhat surprising. We don't know it's true because we don't know what expressions like $e^{\sqrt{2}}$ and e^{π} mean! We have never defined these expressions in school; we've always just behaved as if they probably made sense.

The natural log finally gives us a chance to rectify this oversight. Suppose we define the function $\exp(x)$ to be the inverse function of $\ln(x)$. That is, we say, let $\exp(x)$ be the function defined so that

$$y = \exp x \text{ if and only if } x = \ln y.$$

The functions ln and exp undo one another:

$$\ln(\exp(x)) = x \text{ and } \exp(\ln(x)) = x.$$

Like the graphs of any pair of inverse functions, the graphs of $y = \ln x$ and $y = \exp(x)$ are related by reflection across the line $y = x$. All of this makes sense because $\ln x$ is an increasing function, so that every horizontal line crosses its graph at most once. (This is so because its derivative, $\frac{d}{dx}\ln x = \frac{1}{x} > 0$ for all $x > 0$.)

What can we now say about the function $y = \exp(x)$? Well, if $x = \frac{p}{q}$ is rational, then we know that

$$\ln\left(\exp\left(\frac{p}{q}\right)\right) = \frac{p}{q} = \ln(e^{p/q}).$$

Since ln is an increasing function, this can only happen if

$$\exp\left(\frac{p}{q}\right) = e^{p/q}.$$

We also know that $\ln x$ is a continuous function defined at every $x > 0$, and that its range is the set of all real numbers. This means that its inverse function $\exp(x)$ must also be continuous, and that $\exp(x)$ is defined at every real number x. So $\exp(x)$ is a continuous function defined at every x and equal to e^x at every point at which e^x is defined. So why not do the natural thing here? Why not define

$$e^x = \exp(x)$$

for every real number x? We haven't changed the value of e^x anywhere, but we've now given e^x a value at all those irrational points where it currently lacks one. This is good.

6.4.3 The derivative of the exponential function.

There are now two approaches to computing the derivative of the function $y = e^x$. The first is to be algebraically clever, and to reason like this: We remember that $\ln x$ is an antiderivative of $\frac{1}{x}$, so that

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

We also know that

$$\ln(e^x) = x.$$

Taking the derivative of both sides using the chain rule gives

$$\begin{aligned} \frac{1}{e^x}\cdot\frac{d}{dx}(e^x) &= 1 \\ \frac{d}{dx}(e^x) &= e^x. \end{aligned}$$

A pretty wild result: e^x is its own derivative! This makes sense if you think about the graph of $y = e^x$, shown in Figure 5.

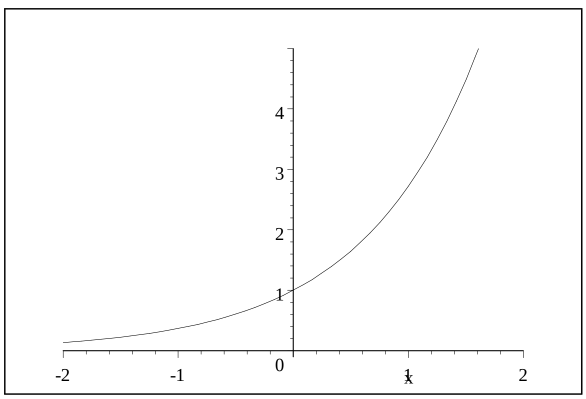

Figure 5: $y = e^x$.

The slope of the tangent line to this function is close to zero for negative x and grows rapidly for positive x, the same as the function itself.

A second approach to computing the derivative of the exponential function is to consider together the graphs of $y = e^x$ and $y = \ln x$, which are shown in Figure 6.

The point (x, e^x) on the graph of $y = e^x$ can be reflected across the line $y = x$ to yield a point (e^x, x) on the graph of $y = \ln x$. The tangent line to the graph of $y = \ln x$ at this point has slope

$$y'(e^x) = \frac{1}{e^x}.$$

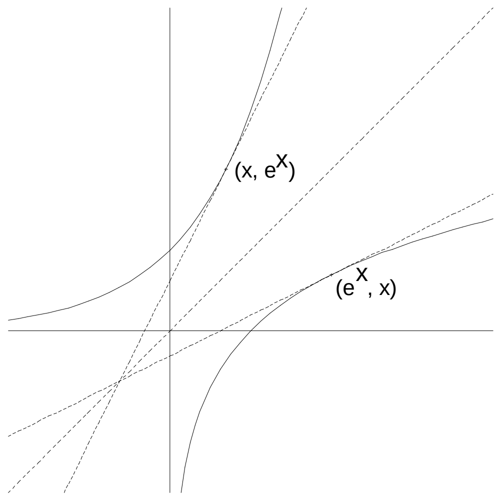

Figure 6: $y = e^x$ and $y = \ln x$.

It is also clear geometrically that the slope of the tangent line to $y = e^x$ at the point (x, e^x) is the reciprocal of the slope of the tangent line to $y = \ln x$ at the point (e^x, x). In algebra,

$$\frac{d}{dx}(e^x) = \frac{1}{1/e^x} = e^x,$$

the same result we got before.

6.4.4 A celebration.

This stuff has all been technical and subtle, but we've made a fantastic amount of progress in a few pages. We've managed to compute an important integral we couldn't compute before. We've defined at least e^x for all the infinitely many irrational numbers—in fact, the overwhelming majority of all real numbers—for which it was previously undefined. We've successfully differentiated both $\ln x$ and e^x. And we've found a function that's its own derivative—the sort of thing there ought to be a use for.

Let's celebrate.

6.5 Other Bases for Logs and Exponential Functions

We've gotten all teary eyed about being able to define and do calculus on the functions e^x and $\ln x$, but what about other exponential and log functions? After all, we didn't care about (or even know about) e^x and $\ln x$ until recently. What about 10^x and $\log_{10} x$, or 2^x and $\log_2 x$?

The answers turn out to be quite simple. As long as $x = \frac{p}{q}$ is rational, so that everything is defined, we can write

$$2^x = (e^{\ln 2})^x = e^{x \ln 2}.$$

When x is irrational, we don't yet have a definition for 2^x. So why not do the natural thing and define

$$2^x = e^{x \ln 2}$$

for all x? We won't have changed the definition of 2^x at any point where it is currently defined, and we'll have ended up with a continuous function we know how to differentiate. After all, the chain rule tells us that

$$\begin{aligned} \frac{d}{dx}(2^x) &= \frac{d}{dx}(e^{x \ln 2}) = \frac{d}{dx}(\exp(x \ln 2)) = \exp(x \ln 2) \cdot \frac{d}{dx}(x \ln 2) \\ &= \exp(x \ln 2) \cdot \ln 2 = e^{x \ln 2} \ln 2 = 2^x \ln 2. \end{aligned}$$

The same thing is true for every other positive constant a.

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

We can do something similar with logarithms. The condition that $y = 2^x$ should be equivalent to $x = \log_2 y$. But it is also equivalent to $y = e^{x \ln 2}$, which is equivalent to $\ln y = x \ln 2$, or to $x = \frac{\ln y}{\ln 2}$. It would therefore be sensible to define

$$\log_2 y = \frac{\ln y}{\ln 2},$$

which would mean that

$$\frac{d}{dy}(\log_2 y) = \frac{d}{dy}\left(\frac{\ln y}{\ln 2}\right) = \frac{1}{y \ln 2}.$$

The same thing is true of any other positive base, a.

$$\frac{d}{dy}(\log_a y) = \frac{d}{dy}\left(\frac{\ln y}{\ln a}\right) = \frac{1}{y \ln a}.$$

Now we can define and differentiate every exponential or logarithmic function everywhere.

6.6 What's Natural About Natural Logs?

When one first says to people, "The natural log is the log base 2.718281828459...," the usual reaction is one of quiet amusement. "Wouldn't, say, 10, be even more natural?" one can hear them thinking. If one thinks about it, though, one can by now articulate lots of reasons why $\ln x$ and e^x are functions that mathematicians would almost have to have invented and studied, while the particular choice of 10^x and $\log_{10} x$ reflects what would seem to be contingent evolutionary choices made by our amphibian ancestors.

One way in which e is the simplest base for logarithms and powers is seen in the derivative formulas

$$\begin{aligned} \frac{d}{dx}\ln x &= \frac{1}{x} \\ \frac{d}{dx}e^x &= e^x, \end{aligned}$$

which are simpler than the analogous formulas for any other base,

$$\begin{aligned} \frac{d}{dx}\log_a x &= \frac{1}{x\ln a} \\ \frac{d}{dx}a^x &= a^x \ln a. \end{aligned}$$

Using any other base than e entails carrying around a bunch of factors of an annoying constant, $\ln a$, which aren't present if you use base e. So base e looks simpler and more natural. Not only that, but the annoying constant $\ln a$ you're carrying around is defined in terms of the natural log, which means that even if you didn't start out interested in natural logs, you'd soon be forced to invent them in order to understand how that constant depended on a. So it looks like you can't deal with derivatives of logs or powers at all without eventually inventing the natural log.

One way of saying this simply by thinking about exponential functions is to say that for any a, the function a^x is a function proportional to its own derivative. Confronted with a whole raft of these functions with different constants of proportionality, wouldn't one naturally look for the one where the constant of proportionality was 1, i.e., the function that was equal to its own derivative? This function that it would be natural to study is e^x.

Of course, one could also argue that after finding antiderivatives for every power of x except for x^{-1}, it would be pretty natural to look for an antiderivative for x^{-1}, which would probably end up with us inventing $\ln x$ for this purpose alone.

In short, even though there is no way to claim that e is the first number a naive person would pick out as a base for logarithms, the important functional properties of $\ln x$ and e^x would seem to make their invention inevitable in any society developing calculus. No other base has this property, so in a very real sense, $\ln x$ is *the* natural logarithm.

6.7 A Glimpse of Differential Equations and Exponential Functions.

The fact that the derivative of e^x is e^x or that the derivative of a^x is $a^x \ln a$, a constant multiple of a^x, makes exponential functions very important in a whole range of applications. If y is a function of time representing the population of a species during good times, or the amount of money in a savings account in good times, or the amount of a radioactive material that has not yet decayed, then one thing we know about y is that its derivative is proportional to itself:

$$y' = ry.$$

The constant r is the growth rate of the population, or the interest rate of the account, or the decay constant of the radioactive material.

There are fancier ways to arrive at this conclusion, but one way to solve this differential equation is just to guess. The function $y = e^{rt}$ solves $y' = ry$. So does

$$y = Ce^{rt}$$

for every constant C. With a small amount of work, one can show that these are the only solutions to this DE. Remember that you saw it here first.

6.8 A final bit of amusement: x^x.

Let a be a constant. For a long time, we've known how to differentiate x^a.

$$\frac{d}{dx}(x^a) = ax^{a-1}.$$

We've just now learned as well how to differentiate a^x.

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

OK, so what about x^x? What's its derivative?

2 wrong answers... Maybe x^x is like x^a, in which case $\frac{d}{dx}(x^x) = xx^{x-1} = x^x$.
Or maybe x^x is like a^x, in which case $\frac{d}{dx}(x^x) = x^x \ln x$.

...and a right one. By the definition we've given for the exponential function,

$$x^x = e^{x \ln x} = \exp(x \ln x).$$

Its derivative is therefore

$$\begin{aligned}
\frac{d}{dx}(e^{x\ln x}) &= \left(e^{x \ln x}\right)\left(\frac{d}{dx}(x \ln x)\right) \\
&= \left(e^{x \ln x}\right)\left(\ln x + x\frac{1}{x}\right) \\
&= \left(e^{x \ln x}\right)(1 + \ln x) \\
&= x^x(1 + \ln x) \\
&= x^x + x^x \ln x.
\end{aligned}$$

The right answer is the sum of the two wrong ones. It's a pity this doesn't work more generally!