7 What's Next?

I want to end this course with just a quick glimpse—but, I hope, a tantalizing one—of what comes next, of where one might go from here.

One way to think about the whole project of finding tangent lines that has occupied us this semester is to say that we've been trying to find the straight line that best approximates $y = f(x)$ near the point $(a, f(a))$. If, for convenience, we take $a = 0$, then we would be looking for a line whose value at 0 matched the value $f(0)$ of f at 0, and whose derivative at 0 matched the derivative $f'(0)$ of f at 0. The equation of this line would be

$$y = f(0) + f'(0)x.$$

Now let's ask the next question: Instead of a straight line, why not take a parabola? What's the parabola that best approximates $y = f(x)$ near $x = 0$?

Let our parabola look like

$$y = a + bx + cx^2.$$

We would certainly want $y(0) = f(0)$, which means that $a = f(0)$. We would also want $y'(0) = f'(0)$, which would mean that $b = f'(0)$. But now we could also ask that $y''(0) = f''(0)$, which would mean that $2c = f''(0)$, or that $c = \frac{f''(0)}{2}$. The parabola therefore looks like

$$y = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

Notice that this is just our best fit straight line with one extra term added on.

It's not hard to think what the next step might be. What if we look for the cubic polynomial that best approximates $f(x)$ near $x = 0$? If we let

$$y = a + bx + cx^2 + dx^3,$$

then a, b, and c would be computed the same way as above, and the results would be the same. Now, however, we could also ask that $y'''(0) = f'''(0)$, which would say that $6d = f'''(0)$, or that $d = \frac{f'''(0)}{6}$. The cubic curve is therefore

$$y = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.$$

If you keep playing this game, using a fourth degree polynomial and trying to get the right 4th derivative at 0, you get

$$y = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4,$$

an so on.

A bit of reflection shows that all the terms we've computed have the same form. The polynomial we just computed can be written as

$$y = f(0) + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4.$$

One can even write the first term in the same form, so as to get

$$y = \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4,$$

as long as we remember that $x^0 = 1$ and that $f^{(0)}$ just means f, and that this fomula, like many others, works out nicely if we define 0! to be 1.

It doesn't take much imagination to see how to continue this sum to get more terms.

The polynomials we're computing this way are called Taylor polynomials, after Brook Taylor, an English mathematician the generation after Newton. They were also known in India to Madahva in the 14th century.

As a concrete example, it's not hard to work out these polynomials explicitly for the function $f(x) = \sin x$. For this function we have

$$\begin{aligned} f^{(0)}(0) &= \sin 0 = 0 \\ f^{(1)}(0) &= \cos 0 = 1 \\ f^{(2)}(0) &= -\sin 0 = 0 \\ f^{(3)}(0) &= -\cos 0 = -1 \\ f^{(4)}(0) &= \sin 0 = 0 \end{aligned}$$

and so on. The sequence of derivatives repeats with period 4. The first few Taylor polynomials for $f(x) = \sin x$ therefore include

$$\begin{aligned} y &= \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 \\ &= x \\ y &= \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \\ &= x - \frac{x^3}{3!} \\ y &= \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!}. \end{aligned}$$

These polynomials are plotted alongside $f(x) = \sin x$ in Figures 1, 2, 3, resp.

Notice that in addition to matching the sine function better and better right at the origin, these polynomials agree with the sine function over bigger and bigger intervals centered at the origin.

The same thing works for $\cos x$, except that $\cos x$ contains even powered terms instead of odd powered terms.

Well, the next thought is clear, right? What if instead of adding up a finite number of terms to get a Taylor polynomial, we just kept adding forever to get a Taylor series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

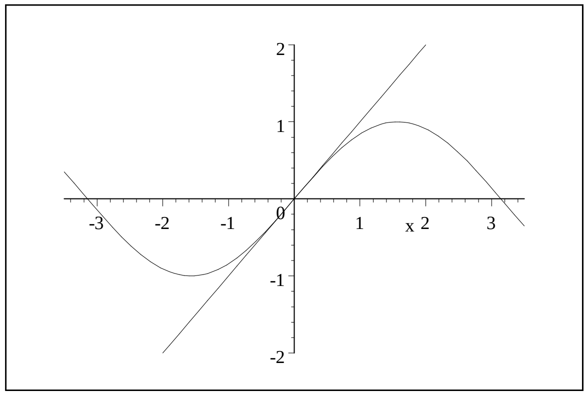

Figure 1: $\sin x$ and its degree 1 Taylor polynomial.

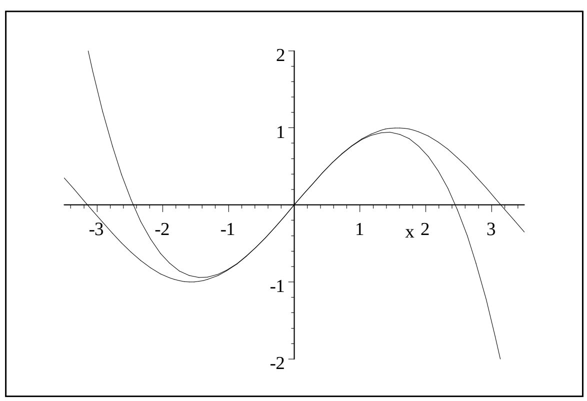

Figure 2: $\sin x$ and its degree 3 Taylor polynomial.

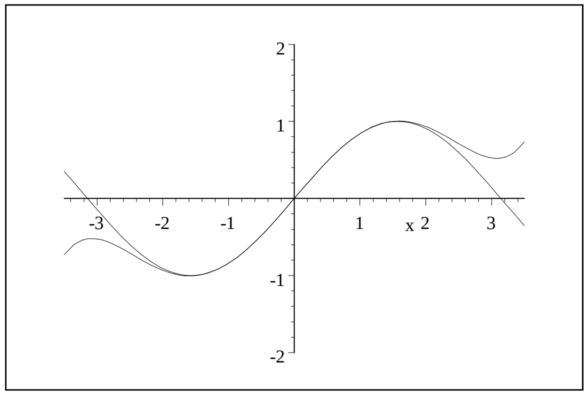

Figure 3: $\sin x$ and its degree 5 Taylor polynomial.

It turns out that in the case of the sine and cosine functions, this infinite series is actually equal to the function everywhere. In other words, for every real number x,

$$\begin{aligned}\sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - ... \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - ...\end{aligned}$$

These formulas, together with some trig identities, are actually what your calculator uses to compute trig functions.

Taylor series give an even simpler expression for the function e^x. We know that if $f(x) = e^x$, then $f'(x) = e^x = f(x)$. For the same reason, for every non-negative integer n, $f^{(n)}(x) = f(x)$. Since $f(0) = e^0 = 1$, it follows that for every non-negative integer n, $f^{(n)}(0) = 1$. The Taylor series for e^x is thus

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$$

Again, it turns out that the equality is exact for all real numbers x.

These series representations prove to be very powerful tools for studying functions. Want to be convinced that $\frac{d}{dx} \sin x = \cos x$? Just take the derivative of the Taylor series for $\sin x$, term by term, and you get the Taylor series for $\cos x$. Want to compute the limit $\lim_{x\to 0} \frac{\sin x}{x}$? Use the Taylor series.

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x}{x} &= \lim_{x\to 0} \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ...}{x} \\ &= \lim_{x\to 0} \left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + ...\right] = 1.\end{aligned}$$

So despite what I said earlier, you can factor an x out of $\sin x$.

I don't want to imply that everything I've just said is trivial. There are in fact a lot of subtle questions that arise here. When does an infinite series converge? When is the Taylor series equal to the original function? When is it legitimate to compute the derivative of a Taylor series by differentiating each term separately? A ton of interesting questions in mathematics are raised by these examples, and resolving them leads to many deep and beautiful discoveries.

A final teaser for the future. A famous theorem of Euler says that if $i = \sqrt{-1}$, then

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

For instance, if $\theta = \pi$, then

$$e^{i\pi} = \cos\pi + i\sin\pi = -1,$$

or, rewriting slightly,

$$e^{i\pi} + 1 = 0.$$

It's been said wryly that this formula is the most beautiful in all mathematics, in part because it contains the 5 most important numbers in mathematics, once each.

What possible sense can one make of Euler's Theorem, though? How do you multiply e by itself an imaginary number of times? How can the claim that $e^{i\theta} = \cos\theta + i\sin\theta$ possibly be a theorem and not, as I thought in high school, some sort of cock-eyed definition?

Well, now that we have formulas for the trig and exponential functions in terms of these infinite series, we could apply those formulas to all possible values of x, even if x is complex. In particular, multiplying things out and then grouping the real and imaginary terms gives

$$\begin{aligned} e^{i\theta} &= 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + ... \\ &= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + ... \\ &= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...\right] + i\left[\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...\right] \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

So Euler's Theorem really is some sort of a theorem, at least if we can somehow justify all the steps in this plausible but hardly proven calculation.

This is only the tiniest introduction to the ideas of infinite series—a major part of mathematics throughout history—and to the idea of doing calculus on functions of the complex numbers. This latter idea gives rise to astoundingly beautiful results, and a good case can be made that what the 19th century was about (at least, in mathematics) was learning how to do all the things we do in calculus to the new functions of a complex variable. The resulting theory has deep and lovely implications in some of the most unexpected places. For instance, it turns out that the best estimates of the distribution of prime numbers arises from the study of the zeros of the Riemann zeta function on the complex plane, hardly the sort of thing one would have expected.

In short, do come back for more. It only gets better—deeper and more beautiful—from here.