REMARKS ON CHALLENGE PROBLEM I

Tim McLarnan

Challenge Problem I was the trig identity from Whittaker and Watson: For every integer $n \geq 2$,

$$\left(\sin \frac{\pi}{n}\right)\left(\sin \frac{2\pi}{n}\right)\dots\left(\sin \frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

I haven't received any solutions to this problem, though I know some folks have been working on it. For those of you who are ready to give up, here's my solution.

To keep things neater, let's write the identity using a product sign, which is just like $\sum$ notation except it tells you to multiply. Written this way, the identity becomes

$$\prod_{r=1}^{n-1}\left(\sin \frac{\pi ir}{n}\right) = \frac{n}{2^{n-1}}.$$

My proof of this identity makes heavy use of a miraculous formula of Euler,

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1}$$

which in turn is proved by plugging $x = i\theta$ into the Taylor series for e^x and then separating out the real and imaginary terms from the result.

Consequences of Euler's theorem include things like

$$e^{\pi i/2} = i \tag{2}$$

and the essential identity

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{3}$$

A final starting fact we'll want is that there are n n^{th} roots of 1 in the complex plane. Geometrically, these should be n points spaced evenly around the unit circle, since multiplying complex numbers can be done geometrically by multiplying their distances from the origin and adding the angles they make with the x-axis. Because of Formula (1), these n points can be written as

$$1,\ e^{2\pi i/n},\ e^{4\pi i/n},\ \dots\ ,\ e^{(n-1)\pi i/n},$$

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or, equivalently, as

$$1,\ e^{-2\pi i/n},\ e^{-4\pi i/n},\ \ldots\ ,\ e^{-(n-1)\pi i/n}. \tag{4}$$

Now we're ready. The left side of Whittaker and Watson's identity can be rewritten as

$$\begin{aligned}
\prod_{r=1}^{n-1}\left(\sin\frac{\pi i r}{n}\right) &= \prod_{r=1}^{n-1}\frac{e^{\pi i r/n}-e^{-\pi i r/n}}{2i} \\
&= \frac{1}{2^{n-1}i^{n-1}}\prod_{r=1}^{n-1}\left(e^{\pi i r/n}-e^{-\pi i r/n}\right) \\
&= \frac{1}{2^{n-1}i^{n-1}}\prod_{r=1}^{n-1}\left(e^{\pi i r/n}\left(1-e^{-2\pi i r/n}\right)\right) \\
&= \frac{e^{\frac{\pi i}{n}(1+2+3+\cdots+(n-1))}}{2^{n-1}i^{n-1}}\prod_{r=1}^{n-1}\left(1-e^{-2\pi i r/n}\right) \\
&= \frac{e^{\frac{\pi i n(n-1)}{2n}}}{2^{n-1}i^{n-1}}\prod_{r=1}^{n-1}\left(1-e^{-2\pi i r/n}\right) \\
&= \frac{\left(e^{\frac{\pi i}{2}}\right)^{n-1}}{2^{n-1}i^{n-1}}\prod_{r=1}^{n-1}\left(1-e^{-2\pi i r/n}\right) \\
&= \frac{1}{2^{n-1}}\prod_{r=1}^{n-1}\left(1-e^{-2\pi i r/n}\right),
\end{aligned}$$

where the last equality comes from formula (2).

Now look at the expression inside the product sign. It looks like the product of 1 minus each of the n^{th} roots of 1 in formula (5) except for 1 itself. Now, the n^{th} roots of 1 must appear in the factorization of $x^n - 1$ over $\mathbb{C}$ like this:

$$x^n - 1 = (x-1)\prod_{r=1}^{n-1}\left(x - e^{-2\pi i r/n}\right).$$

This means that

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = \frac{x^n - 1}{x-1} = \prod_{r=1}^{n-1}\left(x - e^{-2\pi i r/n}\right).$$

Plug in $x = 1$ and lo and behold, you have the product we're trying to evaluate:

$$\prod_{r=1}^{n-1}\left(1 - e^{-2\pi i r/n}\right) = 1^{n-1} + 1^{n-2} + \cdots + 1 + 1 = n.$$

Thus,

$$\prod_{r=1}^{n-1}\left(\sin\frac{\pi ir}{n}\right)=\frac{1}{2^{n-1}}\prod_{r=1}^{n-1}\left(1-e^{-2\pi ir/n}\right)=\frac{n}{2^{n-1}},$$

and we're done.

You might think this argument would be the first thing to occur to me, but, strangely, it was not. Instead, I was slowed down by some interesting mistakes.

My first error was to try to do the whole argument over $\mathbb{R}$ by using clever trig identities, instead of turning everything into complex exponentials. I'd still like a purely real argument for the identity, even though I know it may not be very natural. If you find a completely real argument, by all means tell me about it.

My more critical second error was to try to proceed by induction on n. I wandered into this by discovering that the identity for $2n$ could be reduced by simple real arguments to the identity for n. To do this reduction, notice that the double angle formula for sin gives

$$\left(\sin\frac{\pi r}{2n}\right)\left(\sin\frac{\pi(n+r)}{2n}\right)=\left(\sin\frac{\pi r}{2n}\right)\left(\cos\frac{\pi r}{2n}\right)=\frac{1}{2}\sin\frac{\pi r}{n}.$$

This means that if identity (1) holds for n, then

$$\prod_{r=1}^{2n-1}\left(\sin\frac{\pi ir}{2n}\right)=\frac{1}{2^{n-1}}\prod_{r=1}^{n-1}\left(\sin\frac{\pi ir}{n}\right)=\frac{1}{2^{n-1}}\frac{n}{2^{n-1}}=\frac{2n}{2^{2n-1}},$$

so the identity holds for $2n$.

This reduction occured to me rather quickly, and left me thinking, "Great! We know the theorem for even n, and all we need to do is find some similar reduction for odd n and we'll be done." I therefore wasted an embarrassing amount of time trying to come up with the right combination of uses of the double-angle formula and the half-angle formula to reduce, say, the $n = 11$ case to $n = 5$ or $n = 6$ or both.

On reflection, this was nuts. No possible piece of simple algebraic messing about is going to turn 11^{ths} into 5^{ths} or 6^{ths}. I should have seen this immediately and moved on, but, perhaps beguiled by my partial success, I held on far too long to an obviously fatally flawed program. In some way the lesson of my attempts to prove (1) is just this—don't give too much weight to a partial solution to a problem if it feels like it's leading you down the wrong track.