EIN MATHEMATISCHES OPFER, SATZ 3 LÖSUNGEN

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Mathematical Offering 3 involved the triangle from the USSR Mathematical Olympiad

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & 1 & 1 & & \\
 & 1 & 2 & 3 & 2 & 1 & \\
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

in which each entry is the sum of the entry directly above it plus the two entries diagonally above it. If the top row is labeled row 0, the next row is row 1, and so on, then the problem initially posed in the Olympiad was to prove that every row below the first row contained an even number.

The only solution I received to this problem was one pencilled on the copy of the problem above the drinking fountain, remarking that the entry adjacent to the middle position in each row was always even. This is not correct, as the triangle below shows. Let me therefore just show you my solutions.

The Olympiad's question turns out to be fairly simple if one just starts out and writes all the entries in the triangle mod 2, like this:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & 1 & 1 & & & & \\
 & & & 1 & 0 & 1 & 0 & 1 & & & \\
 & & 1 & 1 & 0 & 1 & 0 & 1 & 1 & & \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{array} .$$

Now, I can think of at least two ways to argue that each row of this triangle contains at least one 0. My approach was this. Because the central element of the top row is 1, and because the entries in each row are symmetric, the central element of every row must be a 1. Now suppose one row of the triangle contained only 1s. It's easy to work backward to compute the contents of the previous rows of the triangle. They must look like this:

$$\begin{array}{ccccccccccccccc}
 & & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots
\end{array}$$

In every possible position, either the row directly above our row of 1s or the row above it contains a 0. Thus, the central number in one of these two rows must be 0, which is impossible. The only rows that can contain all 1s are therefore rows which do not have two rows above them, namely rows 0 and 1 of the triangle.

The solution in Shklarsky, Chentzov and Yaglom's book of Olympiad problems goes like this. They say, look at the leftmost four entries in each row:

$$\begin{array}{cccccccc}
 & & & & & & 1 & \\
 & & & & & 1 & 1 & 1 \\
 & & & & 1 & 0 & 1 & 0 \\
 & & & 1 & 1 & 0 & 1 & \\
 & & 1 & 0 & 0 & 0 & & \\
 & 1 & 1 & 1 & 0 & & & \\
1 & 0 & 1 & 0 & & & &
\end{array}$$

Each of these entries depends only on elements in the leftmost 4 positions in the previous row; so once the blocks of four numbers begin to repeat, the repetition continues. Thus, the bottom 4 rows here repeat. Since each of these blocks contains a 0, we have proven not only that every row of the triangle after the first contains an even number, but that the first even number in any row occurs among its first 4 entries.

The *Mathematical Offering* posed two more questions about this triangle: what one could say about which rows contained multiples of other numbers than 2 (in particular, which rows contained multiples of 3), and what connections, if any, it had to Pascal's triangle. Let me address the last of these questions first.

There are any number of similarities between this triangle and Pascal's, but many of them come from a common source. The numbers in the n^{th} row of Pascal's triangle are the coefficients of the terms of various degrees in the expansion of $(1+x)^n$. This is easy to prove inductively. It is true by inspection for small values of n. If it is true for n, and if $\binom{n}{k}$ is the k^{th} entry in the n^{th} row of Pascal's triangle, then

$$(1+x)^{n+1} = (1+x)\sum_{k=0}^{n}\binom{n}{k}x^k = \sum_{k=0}^{n}\left(\binom{n}{k}+\binom{n}{k-1}\right)x^k + x^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^k$$

by the Pascal recurrence; the result follows by induction.

Exactly the same analysis can be used to show that the numbers in the n^{th} row of the Olympiad triangle are the coefficients of the terms of various degrees in the expansion of $(1+x+x^2)^n$. From this, it is obvious that the sum of the entries in row n is 3^n, and that their alternating sum is 1, just as it is obvious that for Pascal's triangle, the corresponding values are 2^n and 0.

Unfortunately, this description of the Olympiad triangle does not mean that its entries are binomial coefficients. The k^{th} entry in the n^{th} row of the Olympiad triangle is in fact

$$\sum_{t\le k/2}\binom{n}{t}\binom{n-t}{k-2t},$$

which is not so pleasant to work with.

Now, what about the final question on the presence of multiples of numbers other than 2 in the rows of the Olympiad triangle? Unfortunately, it looks to me as if this question is not too easy in general, though I can answer it for the multiples of 3.

Modulo 3, the first 14 rows of the Olympiad triangle look like this:

$$
\begin{array}{c}
1 \\
1\ \ 1\ \ 1 \\
1\ \ 2\ \ 0\ \ 2\ \ 1 \\
1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1 \\
1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1 \\
1\ \ 2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 2\ \ 1 \\
1\ \ 0\ \ 0\ \ 2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 2\ \ 0\ \ 0\ \ 1 \\
1\ \ 1\ \ 1\ \ 2\ \ 2\ \ 2\ \ 0\ \ 0\ \ 0\ \ 2\ \ 2\ \ 2\ \ 1\ \ 1\ \ 1 \\
1\ \ 2\ \ 0\ \ 1\ \ 2\ \ 0\ \ 1\ \ 2\ \ 0\ \ 2\ \ 1\ \ 0\ \ 2\ \ 1\ \ 0\ \ 2\ \ 1 \\
1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1 \\
1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1 \\
1\ \ 2\ \ 0\ \ 2\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 0\ \ 2\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 0\ \ 2\ \ 1 \\
1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 1 \\
1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1 \\
1\ \ 2\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 2\ \ 1
\end{array}
$$

Obviously, there are no multiples of 3 in rows 0, 1, 4, and 13 of the triangle; there are multiples of 3 in every other row listed. The general theorem is this:

Theorem. *Row n of the Olympiad triangle contains a multiple of 3 iff n does not have the form $(3^s - 1)/2$.*

To establish this theorem, we remember the origin of the triangle in terms of the coefficients of $(1 + x + x^2)^n$, and we prove some lemmas about particular rows. From here on down, all our arithmetic will be happening modulo 3. That is, we'll be working in the ring $\mathbb{Z}_3[x]$ of polynomials whose coeffcients are in the ring $\mathbb{Z}_3$ of integers mod 3.

Lemma. *Row $(3^s - 3)/2$ of the mod 3 Olympiad triangle looks like*

$$1\,0\,0\,1\,0\,0\,1\,0\,0\,\ldots,$$

and row $(3^s - 1)/2$ looks like

$$1\,1\,1\,1\,1\,1\,1\,1\,1\,\ldots.$$

In other words, in $\mathbb{Z}_3[x]$,

$$(1 + x + x^2)^{(3^s - 3)/2} = 1 + x^3 + x^6 + \cdots + x^{3^s - 3} \tag{1}$$

and

$$(1 + x + x^2)^{(3^s - 1)/2} = 1 + x + x^2 + x^3 + \cdots + x^{3^s - 1}. \tag{2}$$

Proof. Both claims are true by inspection for small values of s. Suppose they are true for s, and let's try to prove them for $s + 1$. We have

$$(1 + x + x^2)^{(3^{s+1} - 3)/2} = \left((1 + x + x^2)^3\right)^{(3^s - 1)/2} = (1 + x^3 + x^6)^{(3^s - 1)/2}.$$

By induction hypothesis (2) with x replaced by x^3, this evaluates to

$$1 + x^3 + x^6 + \cdots + x^{3^{s+1} - 3},$$

which proves claim (1) by induction. Claim (2) follows at once either by direct multiplication by $(1+x+x^2)$ or by using the Olympiad triangle recurrence to produce the next row of the triangle. ■

With this lemma, we've proven that rows of the form $(3^s-1)/2$ consist entirely of 1s. These rows therefore do not contain multiples of 3. We still need to show that every other layer does contain some multiple of 3. I'll do this only semi-formally. A pencil and paper (or a spreadsheet) might be a help.

We now know that for any s, row $(3^s-1)/2$ consists entirely of 1s. Row $(3^s+1)/2$ therefore looks like

$$1\ 2\ 0\ 0\ 0\ \ldots\ 0\ 0\ 0\ 2\ 1.$$

As you go down rows, only one zero can disappear from each end of the central block of zeros with each row. This means that the middle entry of any row from row $(3^s+1)/2$ to row 3^s-1 is 0. All these rows therefore contain multiples of 3.

A second lemma is now helpful.

Lemma. *Row 3^s of the mod 3 Olympiad triangle looks like*

$$1\ 0\ 0\ \ldots\ 0\ 0\ 1\ 0\ 0\ \ldots\ 0\ 0\ 1.$$

In other words, in $\mathbb{Z}_3[x]$,

$$(1+x+x^2)^{3^s} = 1 + x^{3^s} + x^{2\cdot 3^s}.$$

The proof of this lemma is an easy induction, which I'll leave to you.

Once we have this lemma, though, we're home free. Just as before, the long rows of zeros in row 3^s of the triangle can only shrink by one position as you move down a row. The result is that there need to be zeros left in the middle of these blocks in every row from row 3^s to row $(3^{s+1}-3)/2$. Taken together with our earlier zeros, we have now guaranteed that every row below row $(3^s-1)/2$ and above row $(3^{s+1}-1)/2$ must contain at least one 0, which completes the proof of the theorem.

It's more than a bit annoying that we have to work so hard for this result. What the situation is for multiples of an arbitrary integer, I'm not sure at all. Even for Pascal's triangle, which is significantly simpler, the question of what the triangle looks like mod s is still being worked out. Two papers that might be of interest on this topic, both by Andrew Granville, appeared in *The American Mathematical Monthly,* one in volume **99**(1992), 318–331, and one in volume **104**(1997), 848–851.