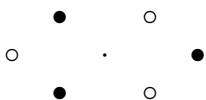


A MATHEMATICAL OFFERING, PROBLEM 4 SOLUTION

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Problem 4 of the Mathematical Offerings was about the pattern



and other *perfect square patterns* (PSP's) in Pascal's triangle. I really loved this problem as an undergraduate because of its nice interplay between algebra, number theory, and geometry. Here's one way to attack it.

To prove that the pattern above is a PSP, suppose that the central empty point in the pattern is $\binom{n}{k}$. Then the entries we are multiplying look like

$$\binom{n}{k-1} \binom{n-1}{k-1} \binom{n-1}{k} \binom{n}{k+1} \binom{n+1}{k} \binom{n+1}{k+1}$$

We need to show that

$$\binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} = \binom{n-1}{k} \binom{n}{k-1} \binom{n+1}{k+1},$$

i.e., that

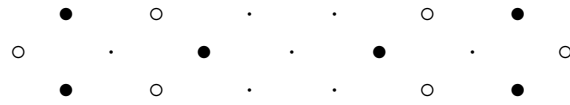
$$\begin{aligned} & \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{n!}{(k+1)!(n-1-k)!} \cdot \frac{(n+1)!}{k!(n+1-k)!} \\ &= \frac{(n-1)!}{k!(n-1-k)!} \cdot \frac{n!}{(k-1)!(n+1-k)!} \cdot \frac{(n+1)!}{(k+1)!(n-k)!}, \end{aligned}$$

which is obvious because the numerators and denominators on both sides contain exactly the same factors in different orders.

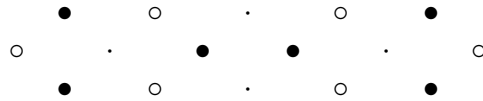
This little hexagonal PSP is going to play an important role in what follows, so it probably needs a name. Let's call it the *elementary hexagon*.

It is possible to use exactly the same machinery to show that the other patterns in the problem are PSP's, but it is more instructive to see if we can't find a general method for

generating PSP's. If we are thinking geometrically, then a natural way to build new PSP's is to take the union of two existing PSP's. For instance, since we already know that the elementary hexagon is a PSP, it follows immediately that patterns like this

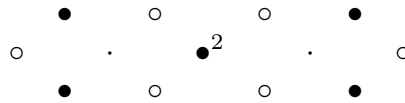


or this



must also be PSP's.

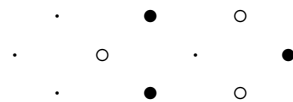
What would happen if we were to slide the two elementary hexagons yet another step closer together so that the two ●'s in the center coalesced? We would still end up with a PSP, as long as we counted the central ● twice:



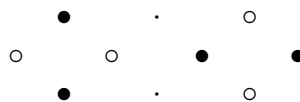
This suggests that we should generalize our definition of a PSP to include patterns of black and white points, each with a multiplicity, in which the product of the black points (each multiplied as many times as its multiplicity) equals the product (with multiplicity) of the white points, regardless of where the pattern is placed in Pascal's triangle. New PSP's can be built from old ones by taking their unions with multiplicity. That is, two black or two white points which are superimposed appear with the sum of their original multiplicities (just as above), while superimposed black and white points cancel one another out. An example of the latter process is the superposition of the elementary hexagon



and the elementary hexagon



which yields the second PSP given in the problem,



I'll leave it as an exercise to obtain the third PSP in the problem as a superposition of copies of the elementary hexagon.

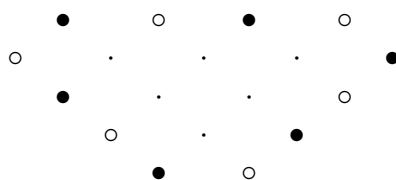
Obviously, the idea of superposition lets one generate effortlessly an infinite number of perfect square patterns. What is by no means so obvious is that in fact, all PSP's are obtained this way.

Another property that all the PSP's we've seen have is that the number of white and black points (counted by multiplicity) are equal in every horizontal row and in every diagonal tier parallel to the edges of Pascal's triangle. Let's call any pattern that satisfies this condition a *balanced pattern*.

Proposition. *Let \mathcal{F} be a figure consisting of white and black points, with multiplicity. The following conditions are equivalent:*

- (1) \mathcal{F} is a PSP.
- (2) \mathcal{F} is a superposition of elementary hexagons.
- (3) \mathcal{F} is balanced.

This proposition, if true, makes it easy to see that the monster,



is a PSP, because it is plainly balanced.

OK, so how do we prove the proposition? Well, as always with this sort of result, some of the implications are easy, and some are hard. We already know that condition (2) implies (1), since the superposition of PSP's is a PSP. It is also easy to prove that (2) \implies (3), since elementary hexagons are balanced, and since the superposition of balanced figures is balanced. We'll complete the proof of the proposition by showing that (1) \implies (3) and that (3) \implies (2).

In order to show that any PSP is balanced, we need to think a bit more about PSPs and about Pascal's triangle. The figure \mathcal{F} is a PSP iff no matter where it is placed in Pascal's triangle, every prime appears to the same power in the product of the black points as in the product of the white points. A useful thing to do might therefore be to investigate geometrically the prime factorizations of the entries in Pascal's triangle.

The first 10 rows of Pascal's triangle look like this:

					1					
					1	1				
				1	2	1				
			1	3	3	1				
		1	4	6	4	1				
	1	5	10	10	5	1				
	1	6	15	20	15	6	1			
	1	7	21	35	35	21	7	1		
	1	8	28	56	70	56	28	8	1	
1	1	9	36	84	126	126	84	36	9	1
1	10	45	120	210	252	210	120	45	10	1

I've circled in green the multiples of the prime 5. Notice how they're arranged: if $p = 5$, then

- (1) The first multiples of p appear in row p of Pascal's triangle (if the top row is row 0). Every element of this row except the 1's on the ends is a multiple of p .
- (2) The multiples of p then form a downward pointing triangle which ends with a single multiple of p in the center of row $2p - 2$.
- (3) Row $2p - 1$ contains no multiple of p .
- (4) Every element of row $2p$ is a multiple of p except the 1's on the ends and the entry $\binom{2p}{p}$ in the very center.
- (5) No multiples of p^2 appear in the first $2p$ rows of the triangle.

It is not hard to show that all these observations hold for every prime $p > 2$. One simply writes the entries in the triangle as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and counts up the multiples of p less than n , k , and $n - k$.

Now we're ready to prove that any PSP is balanced. Let \mathcal{F} be a PSP, and let p be a large prime—big enough that \mathcal{F} can be placed above row p of the triangle, and big enough that \mathcal{F} can be placed entirely within the downward pointing triangle of multiples of p described above. Start with \mathcal{F} sitting above row p and begin sliding it downward. Eventually, the bottom row of \mathcal{F} will reach row p of the triangle. At this stage, every point in the bottom row of \mathcal{F} will be a multiple of p , and no other point in \mathcal{F} will be. If the product of the black points is to equal the product of the white points, then the number of black points in the bottom row must equal the number of white points in that row.

Now slide \mathcal{F} down one row further. The bottom two rows of \mathcal{F} will now be multiples of p , and no other points in \mathcal{F} will be. The number of white points in the bottom two rows of \mathcal{F} must therefore be the number of black points in these rows. Since we already know that the number of white and black points in the bottom row are equal, the same thing must be true of the next to last row.

Continuing this way, we can conclude that in every row of \mathcal{F} , an equal number of black and white points appear.

To deal with the diagonal tiers of \mathcal{F} , just place \mathcal{F} to the left or right of the green triangle of multiples of p , and slide it up into this triangle. The same reasoning as above applies to show that in each diagonal tier parallel to the sides of Pascal's triangle, equal numbers of black and white points appear.

This does it, right? \mathcal{F} is balanced, so our condition (1) \implies (3).

Finally, we show that (3) \implies (2), i.e., that every balanced figure is a superposition of elementary hexagons. To do this, we'll start with a balanced figure \mathcal{F} , and show that by superposing some set S of elementary hexagons on it, we can make \mathcal{F} disappear. This will mean that \mathcal{F} must have been the superposition of hexagons obtained from those in the set S by interchanging the two colors.

Let \mathcal{F} be balanced, and look at its bottom row. By the *length* of this bottom row, I mean the distance between its leftmost and rightmost point in \mathcal{F} . If the leftmost point, P ,

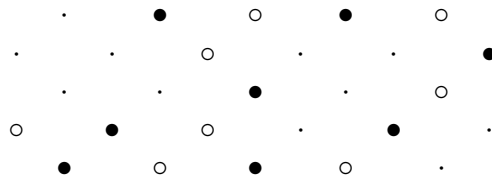
in this row is black (resp., white), then superimpose upon \mathcal{F} a hexagon whose lower left point is a white (resp., black) point situated at P . The result is to replace \mathcal{F} with a new figure missing the bottom left point in \mathcal{F} . (If the point P has multiplicity n , then repeat this superposition n times in order to completely excise P .) The new figure will have a shorter bottom row than \mathcal{F} as long as the bottom row of \mathcal{F} has length greater than 1.

By repeating this process, we can replace \mathcal{F} with a figure \mathcal{F}' whose bottom row has length at most 1. Since \mathcal{F} was balanced, and since the superposition of two balanced figures is balanced, \mathcal{F}' is also balanced. Its bottom row therefore cannot have length 1. This means that the process of superposition that replaced \mathcal{F} with \mathcal{F}' has removed the bottom row of \mathcal{F} completely

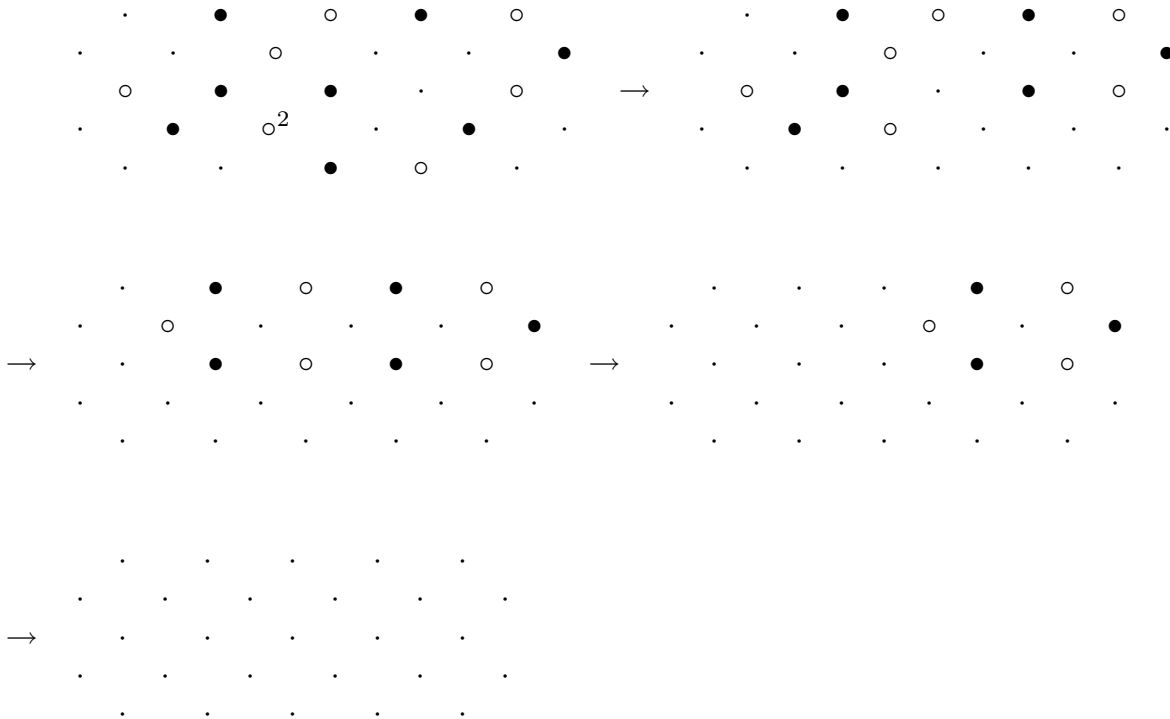
As long as the height of \mathcal{F} was at least 3, the new figure \mathcal{F}' has fewer rows than \mathcal{F} .

Now repeat *this* process, and you will end up with balanced figure of height at most 2. The only such figure is the figure with no points, though, since every non-empty figure of height at most 2 must have a single black or white point in either the first NE-SW diagonal tier to touch the figure or the first NW-SE tier to touch it.

As an illustration of this process of evaporation, suppose we were to start with the balanced figure



Successive superpositions of elementary hexagons transform this figure into



Let me end with some questions to which I do not know the answer, but which I consider to be quite interesting.

- (1) We defined a perfect square pattern not to be a set of black and white points where the product of the blacks equals the product of the whites. Suppose instead we defined it as a collection of gray points for which the product of all the gray numbers was a perfect square. Are there more perfect square patterns with this new definition, or not?
- (2) What about perfect n -power patterns for $n > 2$?
- (3) What can be said about the location of multiples of p (or p^2 , or p^3 , or of a general n) in Pascal's triangle? The references in last week's solutions give an initial set of pointers to current work on this problem.