A MATHEMATICAL OFFERING, PROBLEM 5

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We've been doing discrete problems lately, so let's do one from calculus for a change.

Factorials are important in lots of places in math, but they are also annoying to compute and approximate. (Quick. Which is bigger: 100!, or 10^{150}?) One useful approximation for factorials is

$$e\left(\frac{n}{e}\right)^n < n! < \left(\frac{e}{2}\right)^2\left(\frac{n+1}{e}\right)^{n+1}. \tag{1}$$

Convince me that this approximation is correct.

Hint: $\ln(n!) = \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n)$; and sums are a lot like integrals.

Show that the ratio between the upper and lower bounds for $n!$ in (1) is about $(n+1)e/4$ if n is large. This means our two estimates are still pretty far apart.

Improve these approximations as far as you can. One goal to shoot for (a little tricky, but useful to guide your thinking) is that the ratio of $n!$ to

$$\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}\sqrt{2\pi}$$

approaches 1 as $n \to \infty$.

Tim McLarnan will award a cool prize to the student submitting the best solution of this problem received on or before December 4.

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