A MATHEMATICAL OFFERING, PROBLEM 8: IDENTITY, OR TYPO?

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S. J. Patterson's interesting book, *An Introduction to the Theory of the Riemann Zeta-Function,* mentions a surprising trig identity which is somewhat in the spirit of some of last semester's problems:

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z-n}. \tag{1}$$

Patterson gives two proofs of this identity, and I can think of a third. The more intriguing of these arguments—appealing because it doesn't use any complex analysis or Fourier series—is due to Eisenstein. A central step of Eisenstein's proof, which Patterson passes over without comment, is the following identity:

$$\sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{z^2-m^2} = -2\sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{m^2-n^2}. \tag{2}$$

I initially found identity (2) even more foreign and unfamiliar than identity (1). In fact, I wondered whether there was a typo in Patterson's book.

Was there a typo, or can you prove identity (2)?

As always, I'll award a cool prize to the student submitting the best solution of this problem. You have until Monday, April 5.

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