# A MATHEMATICAL OFFERING, PROBLEM 8 SOLUTION: IDENTITY, OR TYPO?

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Problem 8 asked you to prove or disprove the claim that

$$\sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{z^2-m^2} = -2 \sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{m^2-n^2}.$$

This surprising identity is in fact correct. Not only that, but it's not really all that difficult to prove. A straightforward expansion by partial fractions gives

$$\begin{aligned} \frac{1}{z^2-n^2} \cdot \frac{1}{z^2-m^2} &= \frac{1}{2n(n^2-m^2)(z-n)} - \frac{1}{2n(n^2-m^2)(z+n)} \\ &\quad + \frac{1}{2m(m^2-n^2)(z-m)} - \frac{1}{2m(m^2-n^2)(z+m)} \\ &= \frac{1}{z^2-n^2} \cdot \frac{1}{n^2-m^2} + \frac{1}{z^2-m^2} \cdot \frac{1}{m^2-n^2}. \end{aligned}$$

Summing this over all distinct positive $n$ and $m$ gives

$$\begin{aligned} \sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{z^2-m^2} &= \sum_{n>0} \sum_{0<m\neq n} \frac{1}{z^2-n^2} \cdot \frac{1}{n^2-m^2} + \sum_{n>0} \sum_{0<m\neq n} \frac{1}{z^2-m^2} \cdot \frac{1}{m^2-n^2} \\ &= \sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{n^2-m^2} + \sum_{m>0} \sum_{0<n\neq m} \frac{1}{z^2-m^2} \cdot \frac{1}{m^2-n^2} \\ &= \sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{n^2-m^2} + \sum_{n>0} \sum_{0<m\neq n} \frac{1}{z^2-n^2} \cdot \frac{1}{n^2-m^2} \\ &= \sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{n^2-m^2} + \sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{n^2-m^2} \\ &= -2 \sum_{n>0} \frac{1}{z^2-n^2} \sum_{0<m\neq n} \frac{1}{m^2-n^2}, \end{aligned}$$

which proves the identity.

As long as we're here, let's look at why we are doing this. This identity turned up as part of Eisenstein's proof of a trig identity,

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z-n}. \tag{1}$$

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Eisenstein says, let

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

be the function on the right hand side of this identity. Then

$$\begin{aligned} f(z)^2 &= \frac{1}{z^2} + \sum_{n>0} \left\{ \frac{4z^2}{(z^2-n^2)^2} + \frac{4}{z^2-n^2} \right\} + \sum_{n>0} \sum_{0<m\neq n} \frac{4z^2}{(z^2-n^2)(z^2-m^2)} \\ &= \frac{1}{z^2} + \sum_{n>0} \left\{ \frac{4z^2}{(z^2-n^2)^2} + \frac{4}{z^2-n^2} \right\} - \sum_{n>0} \sum_{0<m\neq n} \frac{8z^2}{(z^2-n^2)(m^2-n^2)} \\ &= \frac{1}{z^2} + \sum_{n>0} \left\{ \frac{4z^2}{(z^2-n^2)^2} + \frac{4}{z^2-n^2} \right\} - \sum_{n>0} \frac{8z^2}{(z^2-n^2)} \sum_{0<m\neq n} \frac{1}{m^2-n^2}. \end{aligned}$$

(This is where we use the identity I challenged you with.)

Now we look at the last bit of this expression:

$$\sum_{0<m\neq n} \frac{1}{m^2-n^2} = \frac{1}{2n} \sum_{0<m\neq n} \left\{ \frac{1}{m-n} - \frac{1}{m+n} \right\}.$$

Expand out ths sum and you get

$$\begin{aligned} \frac{1}{2n}\Big\{ &- \frac{1}{n-1} - \frac{1}{n-2} - \cdots - \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \cdots \\ &- \frac{1}{n+1} - \frac{1}{n-2} - \cdots - \frac{1}{2n-1} - \frac{1}{2n+1} - \frac{1}{2n+2} - \cdots \Big\}, \end{aligned}$$

which reduces, after a blizzard of cancellation, to

$$\sum_{0<m\neq n} \frac{1}{m^2-n^2} = \frac{3}{4n^2},$$

meaning that we've got

$$f(z)^2 = \frac{1}{z^2} + \sum_{n>0} \left\{ \frac{4z^2}{(z^2-n^2)^2} + \frac{4}{z^2-n^2} \right\} - \sum_{n>0} \frac{6z^2}{n^2(z^2-n^2)}.$$

It is easy to compute $f'(z)$ by direct computation; adding it to $f(z)^2$ and doing a bit of simplification yields

$$f(z)^2 + f'(z) = -6 \sum_{n>0} \frac{1}{n^2}.$$

If we let $A = 6\sum(1/n^2)$, then $y = f(z)$ satisfies the DE

$$\frac{dy}{dz} = -y^2 - A.$$

Separating variables gives

$$\frac{dy}{-y^2 - A} = dz.$$

Integrating both sides yields

$$k - z\sqrt{A} = \arctan \frac{y}{\sqrt{A}},$$

i.e.,

$$y = \sqrt{A} \tan(k - z\sqrt{A}).$$

The function $y$ has poles at every integer point, which means that we must have $\sqrt{A} = \pi$. It s also the case that

$$y(0) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{2}{2n+1} = 0,$$

which implies that $k = \pi/2$. This both proves that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and establishes that

$$y(z) = \pi \tan\left(\frac{\pi}{2} - \pi z\right) = \pi \cot(\pi z),$$

which is the trig identity (1) we set out to prove.

A final comment: Identity (1), which tells you that $\pi \cot(\pi z)$ is the sum of all the functions $1/(z-n)$, might remind you of another identity,

$$\begin{aligned} \frac{\sin(\pi z)}{\pi z} &= \prod_{n\neq 0} \left(1 - \frac{z}{n}\right), \\ &= \prod_{n>0} \left(1 - \frac{z^2}{n^2}\right) \end{aligned} \tag{2}$$

which relates the sine function to the product of all functions $1 - z/n$. These identities should seem similar. Taking the log of both sides of (2) gives

$$\log \sin(\pi z) = \log(\pi z) + \sum_{n>0} \log\left(1 - \frac{z^2}{n^2}\right).$$

Differentiating both sides of this identity gives identity (1). Thus, this cotangent identity is just the logarithmic derivative of the infinite product formula for sine! It might be useful to try to reverse this argument and prove the infinite product formula from the cotangent identity, since the product formula is not really easy to prove with complete rigor.