Report on the paper *Counterexamples to the* 0–1 *Conjecture*
by Timothy McLarnan and Gregory Warrington

The 0-1 *Conjecture* states that (1) the off-diagonal coefficients of the Kazhdan-Lusztig realization of the irreducible representations of the symmetric group S_n are all either 0 or 1 (the diagonal coefficients are ± 1). Two stronger statements are (2) all the $\mu(x, y)$ (coefficients of highest possible degree in the Kazhdan-Lusztig polynomial $P_{x,y}$) in type A are equal to 0 or 1 and (3) the graph obtained on every left cell from the edges defined by the left actions of the generators, saturated by application of the Knuth transformations, is in fact the W-graph defined by Kazhdan and Lusztig (this is known as the Lascoux-Schützenberger conjecture.)

The paper by McLarnan and Warrington gives minimal-rank counterexamples to each of these conjectures. The first counterexample to (1) occurs when $n = 16$; for (2), when $n = 10$; for (3), when $n = 14$. These values for n should already give an idea of the difficulty of the problem. Quite a few people had worked on this problem before, either trying to prove the conjecture or trying to find a counterexample, without success — the computations performed by the authors certainly rank amongst the most impressive Kazhdan-Lusztig computations ever!

In fact it is my understanding that McLarnan had found his counterexamples to (1) and (3) already several years ago (a counterexample to (3) was mentioned in a letter from Adriano Garsia as early as 1988!) Their formal publication is certainly most welcome, and should prevent more people from engaging in a doomed attempt to prove the conjecture. It should be mentioned that counterexamples to (3) have also been found by Ochiai and Kako [O-K] using their own computer program; I think this paper should be cited in the references, even though priority is not at issue here, as Ochiai and Kako did not find their example until they heard about the existence of McLarnan's through this referee.

Of course a counterexample to (1) is also a counterexample to (3), as each of the off-diagonal coefficients in the representation matrices is of the form $\mu(x, y)$; however, it may not be a *minimal* counterexample. The two problems are not of the same nature : in the case of (1), one only has to consider pairs of elements lying in the same left cell, whereas in the case of (2) one needs to consider all pairs of elements in the group. Through a clever use of the minimality condition, the first author has succeeded to prune down the number of pairs to be considered from an impossible 20 trillion to a much more manageable number, and in this way has succeeded in finding counterexamples to (1) and (3) which are minimal not only in the sense of the rank, but also in the sense of the length difference, and for a given length difference, in the sense of the length of y. For (2), it would seem that such a systematic search is currently out of reach; so the unearthing of the counterexample to (2) was the result of some clever guesswork by the second author (!) The relatively small value of n in this counterexample comes as an interesting surprise.

In results involving large amounts of computation, as is certainly the case here, there is of course always the issue of reliability (note that this is not different in nature from the reliablility of a complicated proof, but we mathematicians are much better trained for checking proofs than we are for checking computer programs.) In this respect, the authors have done everything that one could ask for, and then some. They have made their code available on the arXiv (this is as it should be — as a matter of principle, the code for computational results relying on machine computations should be made available for examination and trials.) Not being a Java programmer myself, I haven't been able to get Warrington's program to run on my computer so far (although I have seen it run and have no doubt as to its accuracy.) I had no trouble at all compiling McLarnan's C code, and running it according to the instructions in the accompanying README file. This performs the various stages of the filtering process described after Lemma 14 very convincingly. I couldn't make out how to get through Step 3 with the programs provided, but this should be a minor thing. In any case, they have two completely independent computer programs which (are capable of and) agree on the computation of the full Kazhdan-Lusztig polynomial in the counterexample to (1). For counterexample (2), the situation is even better, as the second author has managed to map out the steps of the computation completely, through a careful choice of descents and a systematic use of the powerful "flattening" lemma due to Billey and himself, so that it can in principle be checked by a courageous reader. A further confirmation can be obtained using the `show` command in [dC], which also yields the same polynomial (the counterexample to (1), on the other hand, is far out of the reach of [dC]).

The paper is very clearly and carefully written. If anything, I would say that the authors perhaps haven't done themselves enough justice in giving an idea of the computational difficulties involved, in particular in the search that led to the counterexample to (1), and in the computation by each of them of the corresponding Kazhdan-Lusztig polynomial. For instance, in the filtering process described in section 3, I would have liked to know how many pairs come out of Step 1 for $n = 16$ (the point is that nothing, however trivial, can be done for 20 *trillion* cases on today's computers; apparently most of the cases don't even come up in the process because whole blocks of them are eliminated in one sweep.) Also, I expect that the hardest part in the Kazhdan-Lusztig computations performed by both authors must have been the traversal of the interval $[x, ys]$ in the Bruhat ordering (or more precisely the extraction of the "flush" elements); very little is said on that, maybe because it would make the paper a bit too technical on the computational side ?

The counterexamples presented in this paper should interest a wide audience of representation theorists, combinatorists, algebraic geometers interested in the geometry of Schubert varieties, and computational mathematicians. I warmly recommend the acceptance of this paper for publication in *Representation Theory.*

[O-K] M. Ochiai and F. Kako, Computational constructions of W-graphs corresponding to Hecke algebras $\mathcal{H}(q, n)$ for n up to 15, *Experiment. Math.* **4** (1995), pp. 61–67.

[dC] F. du Cloux, `Coxeter`, demo version of Coxeter3, available from `http://www.desargues.univ-lyon1.fr/home/ducloux/coxeter/coxeter.html`

As a rare occurrence in my career as a referee, I didn't catch a single misprint in the paper! The only quibble I might have is on page 4, line 13, where the sentence starts with "z is right s-flush ..."; starting a sentence with a symbol is unfortunate, particularly a lower-case one.