THE TORTOISE AND HERR CANTOR

The tortoise and the hare decide to continue their athletic competition. In view of the outcome of the first round, the tortoise allows the hare an integer head start of the hare's choice. The hare might, for instance, decide to start 10 meters ahead of the tortoise.

The race now proceeds as follows: The hare begins by writing its current lead in base 2, as $2^3 + 2$. Any exponents are written in base 2 as well, so that the lead ends up expressed as $2^{(2+1)} + 2$. Knowing that it needs to keep moving, the hare advances until its lead is the same expression with every 2 replaced by a 3: $3^{(3+1)} + 3$. The tortoise plods forward a single meter, to reduce the hare's lead to $3^{(3+1)} + 2 = 83$.

Now the tortoise replaces every 3 in its lead with a 4, and the tortoise plods ahead one meter. The hare's new lead is $4^{(4+1)} + 1 = 1025$.

Repeat this process, the hare always writing its lead in the current base, then incrementing every occurrence the base by 1, and the tortoise always moving one unit forward. The separations after next few cycles are

$$5^{(5+1)} = 15,625$$
$$5(6^6) + 5(6^5) + 5(6^4) + 5(6^3) + 5(6^2) + 5(6) + 5 = 279,935$$
$$5(7^7) + 5(7^5) + 5(7^4) + 5(7^3) + 5(7^2) + 5(7) + 4 = 4,215,754$$

Who wins this race? Is it always the same animal?

The answer may surprise you, and the proof is really astounding—This is an example of a problem in discrete mathematics that can be resolved quickly using Cantor's theory of infinite ordinals!

Come see who the winner is, how ordinals work, and whether the refreshments are Hasenpfeffer or Green Turtle soup. Or maybe something else.

Mathophiles, Monday, 3/4/02, 3PM, D209.

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