ABSTRACT ALGEBRA A
HOMEWORK 5 SOLUTIONS

Chapter 5

5-3. Well, yes, the subgroup $\{1, r, s, t\}$ we worked with in the previous chapter.

A more illuminating way to write the elements of this subgroup is as

$$\begin{gathered} e \\ (12)(34) \\ (13)(24) \\ (14)(23). \end{gathered}$$

5-4. No, it doesn't. If it did have a subgroup H of order 6, then by Theorem 5-3, H would have to contain both an element of order 2 and an element of order 3. Since every element of $\mathcal{A}_4$ other than the identity is either a 3-cycle or a product of 2 2-cycles, this means that H would have to contain a 3-cycle (of order 3) and a product of 2 2-cycles (of order 2).

Suppose, therefore, that H contains (123). (If not, just rename the objects being permuted so that $(123) \in H$.) Suppose also that $(12)(34) \in H$. Then H also contains the elements

$$\begin{aligned} (132) &= (123)^{-1} \\ (134) &= (123)(12)(34) \\ (143) &= (134)^{-1} \\ (14)(23) &= (123)(143). \end{aligned}$$

At this point, H already has order at least 7, contradicting our hope that we could build a subgroup of order 6.

Could we get a subgroup with 6 elements by adding either (14)(23) or (13)(24) to H in place of (12)(34)? No, because either of these alternatives consists of a transposition interchanging two adjacent elements of the cycle (123) and a second transposition interchanging 4 and the remaining element of the 3-cycle (123), which means that by symmetry, we could do just the same argument we did above to get too many elements in H.

If you don't buy this argument from symmetry, just do the calculation directly. It isn't hard.

5-9. The first part of this is just a calculation:

$$(234)^{-1}(123456)(234)(123456)^{-1} = (432)(345) = (245).$$

(The first equality comes from what we know about conjugating cycles by arbitrary permutations, or just by direct computation.)

Now suppose σ is a permutation moving a minimal number of elements and containing some cycle of length 4 or longer, possibly together with some other cycles. By renaming the letters if necessary, we can write

$$\sigma = (123\dots x)\tau,$$

where $x \geq 4$ and where τ moves only numbers $y > x$.

Our normal subgroup containing σ would then also contain

$$(234)^{-1}\sigma(234)\sigma^{-1} = (234)^{-1}(123\dots x)\tau(234)\tau^{-1}(123\dots x)^{-1} \tag{1}$$

We know from studying conjugates that $\tau(234)\tau^{-1} = (\tau(2)\,\tau(3)\,\tau(4)) = (234)$, since τ fixes all $y \leq 4 \leq x$. Thus, the permutation in (1) is

$$(234)^{-1}(123\dots x)(234)(123\dots x)^{-1} = (432)(345) = (245),$$

as shown above. This obviously moves fewer elements than the at least 4 elements moved by σ, which contradicts the assumption that σ is minimal.

5-10. Again, the beginning of this is just a simple computation:

$$\begin{aligned}&(126)^{-1}(1234)(5678)(126)[(1234)(5678)]^{-1}\\ &= (621)(237) = (16237).\end{aligned}$$

The more general calculation, which I leave you to verify, shows that if

$$\sigma = (12\dots k)(k, k+1, \dots, 2k)$$

is a product of two k-cycles, then

$$(1, 2, k+2)^{-1}\sigma(1, 2, k+2)\sigma^{-1} = (1, k+2, 2, 3, k+3) \tag{2}$$

is a 5-cycle, which again moves fewer elements than the original σ. This holds as long as $k \geq 3$. Finally, if σ contains more than these two cycles, then an argument just like that given in Problem 9 shows that the remaining cycles do not change the product in equation (2); so that (2) continues to give us an element moving fewer elements than does σ.

5-11. This one is purely computational, but it actually takes two steps. First, do what the Maxfields suggest to show that

$$(321)(12)(34)\cdots(2k-1, 2k)(123)(2k-1, 2k)\cdots(34)(12) = (14)(23).$$

This shows that if N contains a product of disjoint transpositions, then it must contain a product of 2 disjoint transpositions. But $\sigma = (14)(23)$ cannot be move the minimal number of elements moved by elements of N, since a small modification of the Maxfields' proof of *iii* for $\mathcal{A}_5$ shows that if $\sigma = (14)(23) \in N$, then so is

$$(145)^{-1}\sigma(145)\sigma^{-1} = (541)(14)(23)(145)(23)(14) = (145).$$

which moves fewer elements than does σ.

5-13. The proof for $\mathcal{A}_5$ falls conceptually into these steps:

(1) If $\sigma \in N \lhd \mathcal{A}_5$ moves as few elements as possible, then every cycle in σ has the same length.
(2) Since σ is even, σ consists either of a 5-cycle, a product of two 2-cycles, or a 3-cycle.
(3) If σ is a 5-cycle, then N also contains a 3-cycle which moves fewer elements than does σ, so σ isn't minimal. Thus, σ can't be a 5-cycle.

(4) If σ is a product of 2 disjoint transpositions, then N also contains a 3-cycle which moves fewer elements than does σ, so σ isn't minimal. This, σ can't be a product of 2 disjoint transpositions. Thus, σ must be a 3-cycle.
(5) If $N \lhd \mathcal{A}_5$ contains a 3-cycle, then it must contain every 3-cycle.
(6) If $N \lhd \mathcal{A}_5$ contains every 3-cycle, then $N = \mathcal{A}_5$.

The proof of step (1) works without modification in every $\mathcal{A}_n$.

Step (2) is still true in $\mathcal{A}_4$, except that obviously σ can't consist of a 5-cycle since there are only 4 possible elements to permute.

Step (3) is therefore irrelevant, or perhaps, vacuously true.

Step (4) is false in $\mathcal{A}_4$, since the normal subgroup of $\mathcal{A}_4$ we have seen in previous problems consists of the permutations e, (12)(34), (13)(24), and (14)(23). It probably isn't surprising that this step should break, since the Maxfields' proof in $\mathcal{A}_5$ relied on using their formula (1) with $a = (125)$, which we can't do in $\mathcal{A}_4$ because we don't have the element 5 available.

We could really stop here, but it might be worth going on and convincing yourself that the proofs we gave for steps (5) and (6) actually work just fine in $\mathcal{A}_4$. Thus, the only bit of the proof that fails is step (3).