

CALCULUS A
HOMEWORK 3 SOLUTIONS

SECTION 2.1

- 3. (a)** If $P = (1, \frac{1}{2})$ and $Q = (x, x/(1+x))$, then the line joining P and Q has slope

$$\frac{\frac{x}{1+x} - \frac{1}{2}}{x - 1} = \frac{2x - (1+x)}{2(1+x)(x-1)} = \frac{x-1}{2(x^2-1)}.$$

Plugging in the various values of x into *Maple* gives the results in Table 1.

- (b)** Well, the obvious guess would be that the slope equals $\frac{1}{4}$, wouldn't it?
The problem didn't ask you to do this, but you could have worked out the limiting slope exactly:

$$\lim_{x \rightarrow 1} \frac{x-1}{2(x^2-1)} = \lim_{x \rightarrow 1} \frac{1}{2(x+1)} = \frac{1}{4}.$$

- (c)** The tangent line will be the line with slope $\frac{1}{4}$ through the point $(1, \frac{1}{2})$. The equation of this line is $y - \frac{1}{2} = \frac{1}{4}(x - 1)$, or $y = \frac{1}{4}x + \frac{1}{4}$.

As a useful check, the curve and its tangent are shown in Figure 1.

SECTION 2.2

- 5. (a)** -1 .
(b) -2 .
(c) DNE: there are different limits from above and from below.
(d) 2 .
(e) 0 .
(f) DNE: there are different limits from above and from below.
(g) 1 .
(h) 3 .

x	slope
0.5	0.3333333334
0.9	0.2631578950
0.99	0.2512562800
0.999	0.2501251000
1.5	0.2000000000
1.1	0.2380952380
1.01	0.2487562200
1.001	0.2498751000

TABLE 1. Problem 2.1.3: Points of interest.

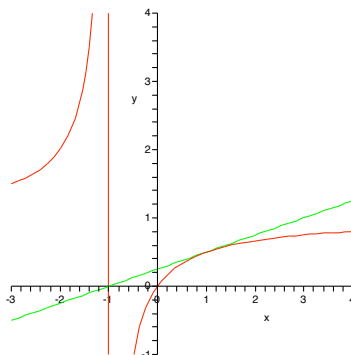


FIGURE 1. Problem 2.1.3: The curve and its tangent.

9. The graph is shown as Figure 2. Zooming in doesn't change the overall picture. It therefore appears that

$$\lim_{x \rightarrow 0^+} f(x) = 0, \quad \lim_{x \rightarrow 0^-} f(x) = 1, \quad \lim_{x \rightarrow 0} f(x) \text{ DNE.}$$

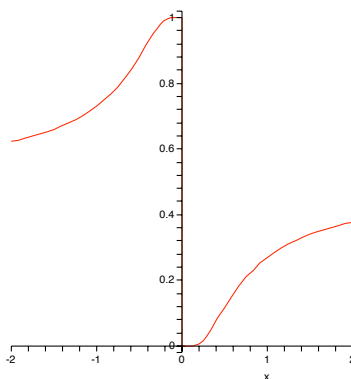


FIGURE 2. Problem 2.2.9: $y = 1/(1 + e^{1/x})$.

It might be interesting to see if you can convince yourself of these facts algebraically. As $x \rightarrow 0^+$, what happens to $1/x$? So what happens to $e^{1/x}$? So what happens to $1/(1 + e^{1/x})$? What if $x \rightarrow 0^-$?

Isn't this a cool function?

14. One example is shown in Figure 3.
19. The numerical data in Table 2 seems strongly to suggest that the limit is 5.
27. Again, it's hard to guess in advance what the solution to this problem is going to be. As $x \rightarrow 0$, $1 + x \rightarrow 1$. Since 1 to any power is 1, you might guess that the

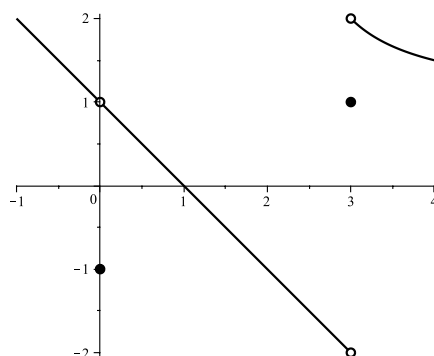


FIGURE 3. Problem 2.2.14: A function with some values and limits.

t	$\frac{e^{5t-1} - 1}{t}$
0.5	22.3649879214069
-0.5	1.83583000275220
0.1	6.48721270700128
-0.1	3.93469340287367
0.01	5.12710963760241
-0.01	4.87705754992860
0.001	5.01252085940096
-0.001	4.98752080731768
0.0001	5.00125020835851
-0.0001	4.99875020830709
0.00001	5.00012500208591
-0.00001	4.99987500208743

TABLE 2. Problem 2.2.19: $\lim_{t \rightarrow 0} \frac{e^{5t} - 1}{t} = 5?$

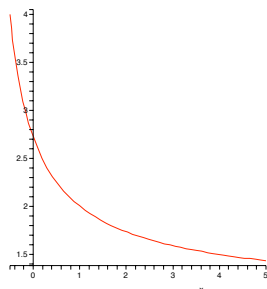
answer would be 1. On the other hand, as $x \rightarrow 0$, the exponent $1 + x \rightarrow \infty$. Any number bigger than 1 raised to an almost infinite power will be huge, so you might guess the result would be $+\infty$.

There is some data addressing this question in Table 3. The first 5 digits of the limit seem to be 2.71828, which sure looks like e . A plot is in Figure 4.

- 28.** One could do this by just plugging in some numbers. I've done this in Table 4. To get these numbers, I've used *Sage* and done the calculations to 100 bit accuracy in order to eliminate most round-off error.

The slope should be somewhere between the last two entries in the table. Our best guess might be that it is something like the average of these two numbers, 0.69314718055994586445831862848.

x	$(1+x)^{1/x}$
1.0	2.0
0.1	2.593742460
0.01	2.704813829
0.001	2.716923932
0.0001	2.718145927
0.00001	2.718268237
0.000001	2.718280469
0.0000001	2.718281692

TABLE 3. Problem 2.2.27: A limit of e ?FIGURE 4. Problem 2.2.27: $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$?

x	$(2^x - 1)/x$
0.1	0.71773462536293164213006325024
-0.1	0.66967008463192584018656733850
0.01	0.69555500567188088326982141139
-0.01	0.69075045629640984667897831120
0.0000001	0.69314720458259656036840206102
-0.0000001	0.69314715653729516854823519594

TABLE 4. Problem 2.2.28: What's this limit?

Now we face a question that often turns up in doing math—what's this number? There are several tools on the Web for attempting to answer this question. One useful one is Plouffe's Inverter, <http://pi.lacim.uqam.ca/eng/>. Putting our number into Plouffe's inverter and browsing in the neighborhood turns up the fact that $0.6931471805599453 = \ln 2$. Could this be our number? Well, if we approximate the slope by using the average of the values at 10^{-10} and at -10^{-10} , we get a slope of 0.6931471805599453094 , and the first 19 digits of $\ln 2$ are ... *drum roll*... 0.6931471805599453094 ! So it looks like the limit is $\ln 2$. Knowing this, it might be possible to think about how to prove something.

3. Just sigh and work your way through it.

$$\begin{aligned} \lim_{x \rightarrow -2} (3x^4 + 2x^2 - x + 1) &= \lim_{x \rightarrow -2} (3x^4) + \lim_{x \rightarrow -2} (2x^2) - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \text{ by laws 1 and 2} \\ &= 3 \lim_{x \rightarrow -2} (x^4) + 2 \lim_{x \rightarrow -2} (x^2) - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \quad \text{by law 3} \\ &= 3[\lim_{x \rightarrow -2} x]^4 + 2[\lim_{x \rightarrow -2} x]^2 - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \quad \text{by law 6} \\ &= 3(-2)^4 + 2(-2)^2 - (-2) + 1 \quad \text{by laws 7 and 8} \\ &= 59. \end{aligned}$$

8. (a) The trouble with the equation $\frac{x^2+x-6}{x-2} = x+3$ is that it's not true for one value of x , namely $x=2$. In that case, the left hand side does not exist, while the right hand side does.

(b) The equation $\lim_{x \rightarrow 2} \frac{x^2+x-6}{x-2} = \lim_{x \rightarrow 2} (x+3)$, on the other hand is perfectly acceptable. Both limits are 5. Remember that $f(a)$ (whether it's defined or not) is completely irrelevant to the calculation of $\lim_{x \rightarrow a} f(x)$.

9. Just factor the numerator:

$$\lim_{x \rightarrow 5} \frac{x^2 - 6x + 5}{x - 5} = \lim_{x \rightarrow 5} \frac{(x-1)(x-5)}{x-5} = \lim_{x \rightarrow 5} (x-1) = 4.$$

10. $\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \rightarrow 4} \frac{(x-4)x}{(x-4)(x+1)} = \lim_{x \rightarrow 4} \frac{x}{x+1} = \frac{4}{5}.$

11. Near $x=5$, the numerator of this fraction is close to 6 and the denominator is close to 0. The function therefore has a vertical asymptote at $x=5$, and the limit as $x \rightarrow 5$ does not exist. It wasn't required as part of the problem, but graphing or plugging in values close to $x=5$ shows that

$$\lim_{x \rightarrow 5^+} \frac{x^2 - 5x + 6}{x - 5} = +\infty, \quad \lim_{x \rightarrow 5^-} \frac{x^2 - 5x + 6}{x - 5} = -\infty.$$

12. $\lim_{x \rightarrow -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3} = \lim_{x \rightarrow -1} \frac{(x+1)(2x+1)}{(x+1)(x-3)} = \lim_{x \rightarrow -1} \frac{2x+1}{x-3} = \frac{1}{4}.$

19. Make the 3-story fraction into a 2-story one, and then eliminate a common factor.

$$\begin{aligned} \lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} &= \lim_{x \rightarrow -4} \frac{\frac{4x}{4} + \frac{4x}{x}}{4x(4+x)} \\ &= \lim_{x \rightarrow -4} \frac{x+4}{4x(4+x)} \\ &= \lim_{x \rightarrow -4} \frac{1}{4x} = -\frac{1}{16}. \end{aligned}$$

25. A graph of the function is shown in Figure 5. A reasonable guess would be that the limit is $2/3$.

From Table 5, it appears that the limit (to four decimal places) is around 0.6667, as we had guessed.

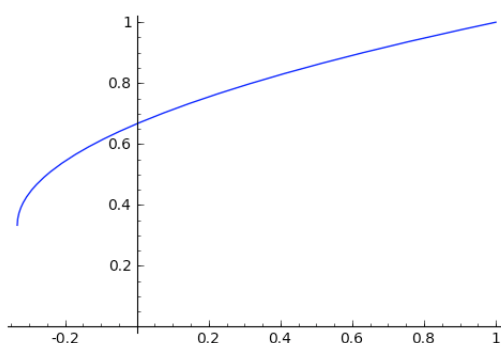


FIGURE 5. Problem 2.3.25: $y = \frac{x}{\sqrt{1+3x-1}}$.

x	$f(x)$
0.1	0.71339
0.01	0.67163
0.001	0.66717
0.0001	0.66672
0.00001	0.66667
-0.1	0.61222
-0.01	0.66163
-0.001	0.66617
-0.0001	0.66662
-0.00001	0.66666

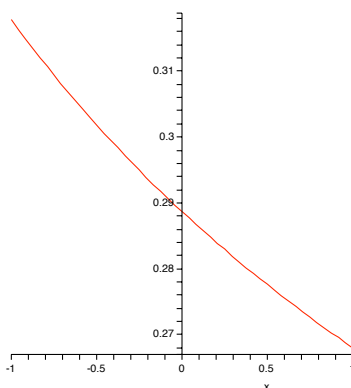
TABLE 5. Problem 2.3.25 Values of $f(x)$ at points near $x = 0$.

To evaluate this limit by an exact calculation, use tricks we've seen before:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+3x}-1} &= \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+3x}-1} \cdot \frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1} \\
 &= \lim_{x \rightarrow 0} \frac{x(\sqrt{1+3x}+1)}{1+3x-1} \\
 &= \lim_{x \rightarrow 0} \frac{x(\sqrt{1+3x}+1)}{3x} \\
 &= \lim_{x \rightarrow 0} \frac{\sqrt{1+3x}+1}{3} = \frac{2}{3}.
 \end{aligned}$$

So we were right.

- 26. (a)** A graph of the function is shown in Figure 6. The limit appears to be around 0.29.
- (b)** Table 6 shows $f(x)$ for various x close to 0. From the table, it appears that the limit (to four decimal places) is around 0.2887.

FIGURE 6. Problem 2.3.26: $y = \frac{\sqrt{3+x}-\sqrt{3}}{x}$.

x	$f(x)$
0.01	0.28843
0.005	0.28855
0.001	0.28865
0.0005	0.28866
0.0001	0.28867
-0.01	0.28892
-0.005	0.28880
-0.001	0.28870
-0.0005	0.28869
-0.0001	0.28868

TABLE 6. Problem 2.3.26 Values of $f(x)$ at points near $x = 0$.

(c) The calculation uses our square root trick:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{3+x} - \sqrt{3})(\sqrt{3+x} + \sqrt{3})}{x(\sqrt{3+x} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 0} \frac{3 + x - 3}{x(\sqrt{3+x} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{3+x} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} \\
 &= \frac{1}{2\sqrt{3}}.
 \end{aligned}$$

The exact value of the limit is $\frac{1}{2\sqrt{3}}$. This agrees with our estimates above, since $\frac{1}{2\sqrt{3}} \approx 0.2886751347$.

- 27.** The value of the cosine is always between 1 and -1 , regardless of the its argument. We therefore have

$$\begin{aligned} -1 &\leq \cos(20\pi x) \leq 1 \\ -x^2 &\leq x^2 \cos(20\pi x) \leq x^2. \end{aligned}$$

It is obvious that $\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} -x^2 = 0$; so be the Squeeze Theorem, $\lim_{x \rightarrow 0} x^2 \cos(20\pi x) = 0$ as well. The plot is shown in Figure 7.

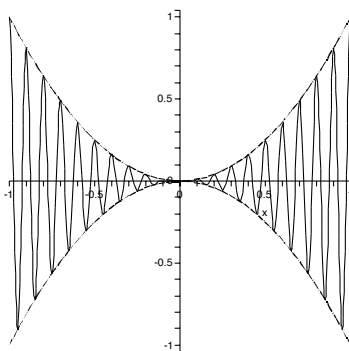


FIGURE 7. Problem 2.3.27: x^2 , $-x^2$, and $x^2 \cos(20\pi x)$.

- 31.** This is just like the previous problem: for every $x \neq 0$, we have

$$\begin{aligned} -1 &\leq \cos \frac{2}{x} \leq 1 \\ -x^4 &\leq x^4 \cos \frac{2}{x} \leq x^4. \end{aligned}$$

Both x^4 and $-x^4$ approach 0 as $x \rightarrow 0$; so be the Squeeze Theorem,

$$\lim_{x \rightarrow 0} x^4 \cos \frac{2}{x} = 0.$$