

**CALCULUS A**  
**HOMEWORK 3 SOLUTIONS**

SECTION 2.1

3. (a) If  $P = (1, \frac{1}{2})$  and  $Q = (x, x/(1+x))$ , then the line joining  $P$  and  $Q$  has slope

$$\frac{\frac{x}{1+x} - \frac{1}{2}}{x - 1} = \frac{2x - (1+x)}{2(1+x)(x-1)} = \frac{x-1}{2(x^2-1)}.$$

Plugging in the various values of  $x$  into *Maple* gives the results in Table 1.

- (b) Well, the obvious guess would be that the slope equals  $\frac{1}{4}$ , wouldn't it?  
The problem didn't ask you to do this, but you could have worked out the limiting slope exactly:

$$\lim_{x \rightarrow 1} \frac{x-1}{2(x^2-1)} = \lim_{x \rightarrow 1} \frac{1}{2(x+1)} = \frac{1}{4}.$$

- (c) The tangent line will be the line with slope  $\frac{1}{4}$  through the point  $(1, \frac{1}{2})$ . The equation of this line is  $y - \frac{1}{2} = \frac{1}{4}(x - 1)$ , or  $y = \frac{1}{4}x + \frac{1}{4}$ .

As a useful check, the curve and its tangent are shown in Figure 1.

SECTION 2.2

5. (a)  $-1$ .  
(b)  $-2$ .  
(c) DNE: there are different limits from above and from below.  
(d)  $2$ .  
(e)  $0$ .  
(f) DNE: there are different limits from above and from below.  
(g)  $1$ .  
(h)  $3$ .

$x$	slope
0.5	0.3333333334
0.9	0.2631578950
0.99	0.2512562800
0.999	0.2501251000
1.5	0.2000000000
1.1	0.2380952380
1.01	0.2487562200
1.001	0.2498751000

TABLE 1. Problem 2.1.3: Points of interest.

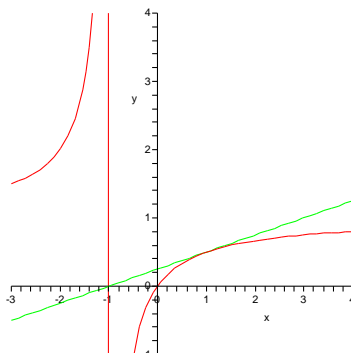


FIGURE 1. Problem 2.1.3: The curve and its tangent.

7. The graph is shown as Figure 2. Zooming in doesn't change the overall picture. It therefore appears that

$$\lim_{x \rightarrow 0^+} f(x) = 0, \quad \lim_{x \rightarrow 0^-} f(x) = 1, \quad \lim_{x \rightarrow 0} f(x) \text{ DNE.}$$

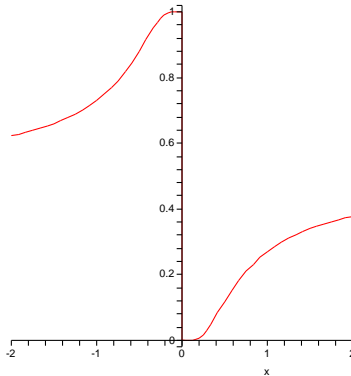


FIGURE 2. Problem 2.2.7:  $y = 1/(1 + e^{1/x})$ .

It might be interesting to see if you can convince yourself of these facts algebraically. As  $x \rightarrow 0^+$ , what happens to  $1/x$ ? So what happens to  $e^{1/x}$ ? So what happens to  $1/(1 + e^{1/x})$ ? What if  $x \rightarrow 0^-$ ?

Isn't this a cool function?

10. There are infinitely many functions that satisfy these constraints. One possible example is in Figure 3.
16. What's interesting about this problem is that as  $x \rightarrow 0^+$ ,  $\ln(x + x^2) \rightarrow -\infty$ , and  $x \rightarrow 0$ . The function therefore seems to approach  $\infty \cdot 0$ , which, like  $0/0$ , could be

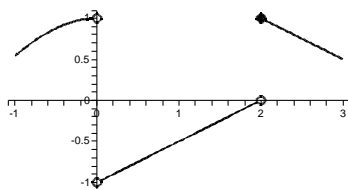


FIGURE 3. Problem 2.2.10: A function with some limits.

anything. If we just plug in values, though, we find

$$\begin{aligned} g(1) &= 0.6931471806 \\ g(0.5) &= -0.1438410362 \\ g(0.1) &= -0.2207274913 \\ g(0.05) &= -0.1473471054 \\ g(0.01) &= -0.04595219855 \\ g(0.005) &= -0.02646664912 \\ g(0.001) &= -0.006906755779 \end{aligned}$$

This sure seems to suggest that the limit is going to be 0. The plot in Figure 4 seems to back this up.

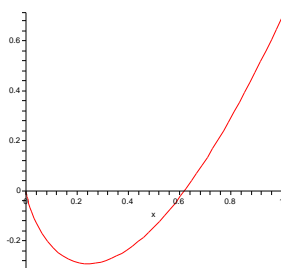


FIGURE 4. Problem 2.2.16:  $y = x \ln(x + x^2)$ .

- 23.** Again, it's hard to guess in advance what the solution to this problem is going to be. As  $x \rightarrow 0$ ,  $1 + x \rightarrow 1$ . Since 1 to any power is 1, you might guess that the answer would be 1. On the other hand, as  $x \rightarrow 0$ , the exponent  $1 + x \rightarrow \infty$ . Any

$x$	$(1+x)^{1/x}$
1.0	2.0
0.1	2.593742460
0.01	2.704813829
0.001	2.716923932
0.0001	2.718145927
0.00001	2.718268237
0.000001	2.718280469
0.0000001	2.718281692

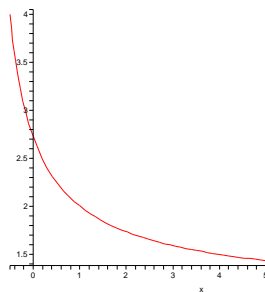
TABLE 2. Problem 2.2.23: A limit of  $e$ ?

$x$	$(2^x - 1)/x$
0.1	0.7177346254
-0.1	0.6696700846
0.01	0.6955550057
-0.01	0.6907504563
0.000001	0.6931474207
-0.000001	0.6931469403

TABLE 3. Problem 2.2.18: What's this limit?

number bigger than 1 raised to an almost infinite power will be huge, so you might guess the result would be  $+\infty$ .

There is some data addressing this question in Table 2. The first 5 digits of the limit seem to be 2.71828, which sure looks like  $e$ . A plot is in Figure 5.

FIGURE 5. Problem 2.2.23:  $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$ ?

- 24.** One could do this by just plugging in some numbers. I've done this in Table 3. To get these numbers, I've used *Maple* and done the calculations to 30 digit accuracy in order to eliminate most round-off error.

The table makes it clear that the slope is somewhere between 0.6931474207 and 0.6931469403. Our best guess might be that it is something like the average of these two numbers, 0.6931471806.

Now we face a question that often turns up in doing math—what’s this number? I gave you a link for the Inverse Symbolic Calculator as one possible tool for addressing this problem. The ISC doesn’t get an exact match for 0.6931471806, but it does tell us that  $0.6931471805599453 = \ln 2$ . Could this be our number? Well, if we approximate the slope by using the average of the values at  $10^{-10}$  and at  $-10^{-10}$ , we get a slope of 0.6931471805599453094, and the first 19 digits of  $\ln 2$  are . . . *drum roll* . . . 0.6931471805599453094! So it looks like the limit is  $\ln 2$ . Knowing this, it might be possible to think about how to prove something.

Incidentally, a less sophisticated way to get the same result might have been to remember that the same number, 0.6931471806, had turned up in the first line of the data in Problem 16 as  $g(1) = 1 \ln(1 + 1^2) = \ln 2$ . Remembering everything you’ve ever seen is clearly a plus in any kind of scholarship.

## SECTION 2.3

3. Just sigh and work your way through it.

$$\begin{aligned} \lim_{x \rightarrow -2} (3x^4 + 2x^2 - x + 1) &= \lim_{x \rightarrow -2} (3x^4) + \lim_{x \rightarrow -2} (2x^2) - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \text{ by laws 1 and 2} \\ &= 3 \lim_{x \rightarrow -2} (x^4) + 2 \lim_{x \rightarrow -2} (x^2) - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \quad \text{by law 3} \\ &= 3[\lim_{x \rightarrow -2} x]^4 + 2[\lim_{x \rightarrow -2} x]^2 - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \quad \text{by law 6} \\ &= 3(-2)^4 + 2(-2)^2 - (-2) + 1 \quad \text{by laws 7 and 8} \\ &= 59. \end{aligned}$$

8. (a) The trouble with the equation  $\frac{x^2+x-6}{x-2} = x+3$  is that it’s not true for one value of  $x$ , namely  $x = 2$ . In that case, the left hand side does not exist, while the right hand side does.

- (b) The equation  $\lim_{x \rightarrow 2} \frac{x^2+x-6}{x-2} = \lim_{x \rightarrow 2} (x+3)$ , on the other hand is perfectly acceptable. Both limits are 5. Remember that  $f(a)$  (whether it’s defined or not) is completely irrelevant to the calculation of  $\lim_{x \rightarrow a} f(x)$ .

9. Look for a common factor and remove it.

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 3)}{x - 2} = \lim_{x \rightarrow 2} (x + 3) = 5.$$

10. Same idea, except you have to factor both the numerator and the denominator

$$\begin{aligned} \lim_{x \rightarrow -4} \frac{x^2 + 5x + 4}{x^2 + 3x - 4} &= \lim_{x \rightarrow -4} \frac{(x + 1)(x + 4)}{(x - 1)(x + 4)} \\ &= \lim_{x \rightarrow -4} \frac{x + 1}{x - 1} \\ &= \frac{-3}{-5} = \frac{3}{5} \end{aligned}$$

11. This time the limit doesn’t exist, since the numerator approaches 8 while the denominator approaches 0.

12. If you had trouble factoring this one, remember that the offending factor had to be  $x - 4$ . Divide this out of the numerator and denominator, and you're on your way.

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 3x - 4} &= \lim_{x \rightarrow 4} \frac{x(x - 4)}{(x + 1)(x - 4)} \\ &= \lim_{x \rightarrow 4} \frac{x}{x + 1} = \frac{4}{5}\end{aligned}$$

19. The tricky part of this one is that you need to multiply both numerator and denominator by  $\sqrt{x + 2} + 3$ . It may seem strange to put a square root into the denominator instead of the numerator, but it gives us exactly the cancellation we need.

$$\begin{aligned}\lim_{x \rightarrow 7} \frac{\sqrt{x + 2} - 3}{x - 7} &= \lim_{x \rightarrow 7} \frac{(\sqrt{x + 2} - 3)(\sqrt{x + 2} + 3)}{(x - 7)(\sqrt{x + 2} + 3)} \\ &= \lim_{x \rightarrow 7} \frac{(x + 2) - 9}{(x - 7)(\sqrt{x + 2} + 3)} \\ &= \lim_{x \rightarrow 7} \frac{x - 7}{(x - 7)(\sqrt{x + 2} + 3)} \\ &= \lim_{x \rightarrow 7} \frac{1}{\sqrt{x + 2} + 3} \\ &= \frac{1}{\sqrt{9} + 3} = \frac{1}{6}\end{aligned}$$

21. Make the 3-story fraction into a 2-story one, and then eliminate a common factor.

$$\begin{aligned}\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} &= \lim_{x \rightarrow -4} \frac{\frac{4x}{4} + \frac{4x}{x}}{4x(4 + x)} \\ &= \lim_{x \rightarrow -4} \frac{x + 4}{4x(4 + x)} \\ &= \lim_{x \rightarrow -4} \frac{1}{4x} = -\frac{1}{16}.\end{aligned}$$

24. (a) A graph of the function is shown in Figure 6. The limit appears to be around 0.29.

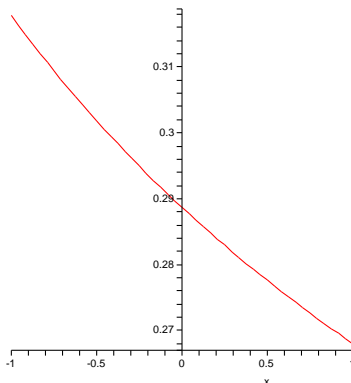


FIGURE 6. Problem 2.3.24:  $y = \frac{\sqrt{3+x} - \sqrt{3}}{x}$

$x$	$f(x)$
0.01	0.28843
0.005	0.28855
0.001	0.28865
0.0005	0.28866
0.0001	0.28867
-0.01	0.28892
-0.005	0.28880
-0.001	0.28870
-0.0005	0.28869
-0.0001	0.28868

TABLE 4. Problem 2.3.24: Values of  $f(x)$  at points near  $x = 0$ .

- (b) Table 4 shows  $f(x)$  for various  $x$  close to 0. From the table, it appears that the limit (to four decimal places) is around 0.2887.
- (c) The calculation uses our square root trick:

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{3+x} - \sqrt{3})(\sqrt{3+x} + \sqrt{3})}{x(\sqrt{3+x} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 0} \frac{3+x-3}{x(\sqrt{3+x} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{3+x} + \sqrt{3})} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} \\
 &= \frac{1}{2\sqrt{3}}.
 \end{aligned}$$

The exact value of the limit is  $\frac{1}{2\sqrt{3}}$ . This agrees with our estimates above, since  $\frac{1}{2\sqrt{3}} \approx 0.2886751347$ .

- 25.** The value of the cosine is always between 1 and  $-1$ , regardless of the its argument. We therefore have

$$\begin{aligned}
 -1 &\leq \cos(20\pi x) \leq 1 \\
 -x^2 &\leq x^2 \cos(20\pi x) \leq x^2.
 \end{aligned}$$

It is obvious that  $\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} -x^2 = 0$ ; so be the Squeeze Theorem,  $\lim_{x \rightarrow 0} x^2 \cos(20\pi x) = 0$  as well. The plot is shown in Figure 7.

- 29.** This is just like the previous problem: for every  $x \neq 0$ , we have

$$\begin{aligned}
 -1 &\leq \cos \frac{2}{x} \leq 1 \\
 -x^4 &\leq x^4 \cos \frac{2}{x} \leq x^4.
 \end{aligned}$$

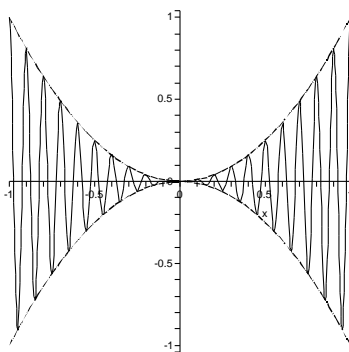


FIGURE 7. Problem 2.3.25:  $x^2$ ,  $-x^2$ , and  $x^2 \cos(20\pi x)$ .

Both  $x^4$  and  $-x^4$  approach 0 as  $x \rightarrow 0$ ; so by the Squeeze Theorem,

$$\lim_{x \rightarrow 0} x^4 \cos \frac{2}{x} = 0.$$