

CALCULUS A
HOMEWORK 5 SOLUTIONS

SECTION 2.7

2. I would guess that

$$f'(0) = 4$$

$$f'(1) = 0$$

$$f'(2) = -1.5$$

$$f'(3) = -1.5$$

$$f'(4) = -1$$

$$f'(5) = -1/3$$

Your estimates will probably be slightly different from mine. The sketch of the derivative should look something like the curve in Figure 1.

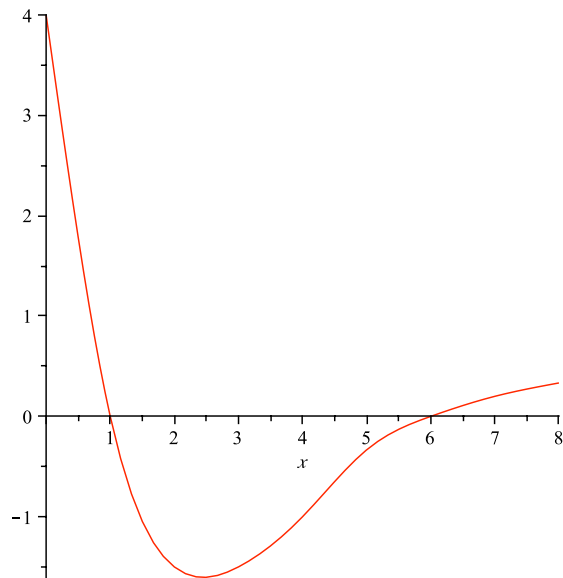


FIGURE 1. Problem 2.7.2: Sketch of $y = f'(x)$.

3. The answers are (a)-II, (b)-IV. (c)-I, (d)-III. Reasons for these choices are a little tedious, basically amounting to saying that if you walk down the x -axis and look at

the slope at each point, you get a curve pretty much like the one I've paired with the function.

5. A rough sketch looks something like that shown in Figure 2.

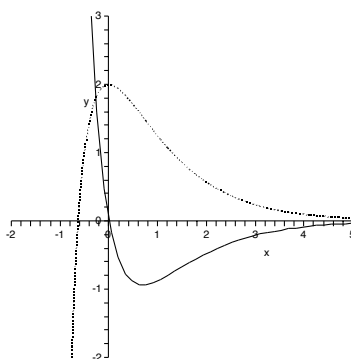


FIGURE 2. Problem 2.7.5: $f(x)$ (dotted) and $f'(x)$ (solid).

7. This time the sketch is in Figure 3.

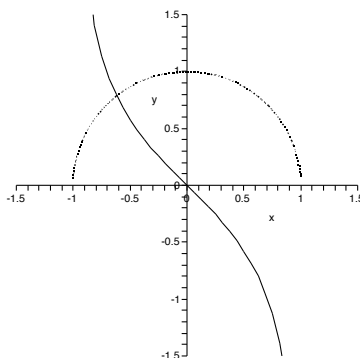
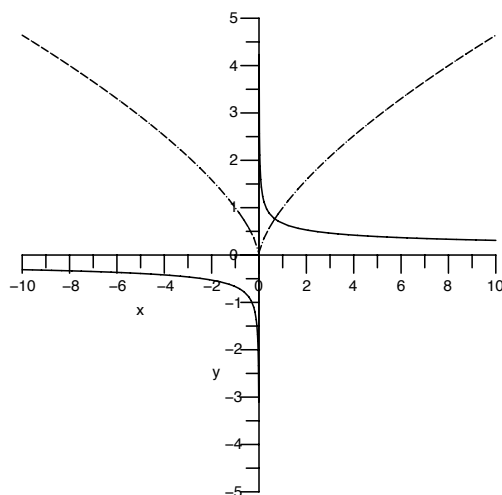


FIGURE 3. Problem 2.7.7: $f(x)$ (dotted) and $f'(x)$ (solid).

8. This time the derivative is discontinuous, as shown in Figure 4. I've drawn this assuming the slope of the function is supposed to become infinite near the origin, but one really can't tell that for sure from the graph in the problem.

FIGURE 4. Problem 2.7.8: $f(x)$ (dotted) and $f'(x)$ (solid).

12. The graph should be a sort of bell-shaped curve with horizontal asymptotes at height 0 in both directions and with a maximum near $t = 6$, when the yeast population is growing most rapidly.

18. (a) Good estimates would be

$$\begin{aligned} f'(0) &= 0 \\ f'\left(\frac{1}{2}\right) &= \frac{3}{4} \\ f'(1) &= 3 \\ f'(2) &= 12 \\ f'(3) &= 27. \end{aligned}$$

Your estimates may vary slightly from these, but hopefully they are pretty close.

(b) From the graph, it looks like $f'(-x) = f'(x)$; so we expect

$$\begin{aligned} f'\left(-\frac{1}{2}\right) &= \frac{3}{4} \\ f'(-1) &= 3 \\ f'(-2) &= 12 \\ f'(-3) &= 27. \end{aligned}$$

(c) I'll let you sketch the graph.

(d) A reasonable guess would be $f'(x) = 3x^2$.

t	$P'(t)$	t	$P'(t)$
1950	0.46	1980	-0.415
1960	0.145	1990	-0.115
1970	-0.385	2000	0

TABLE 1. Problem 2.8.30: $P'(t)$.

(e) One way to do it is like this:

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2.
 \end{aligned}$$

34. (a) $P'(t)$ has units percent/year and measures the annual change in the percentage of Americans under age 18.

(b) Probably the best way to get $P'(1970)$ would be to take the average

$$\frac{1}{2} \left\{ \frac{P(1980) - P(1970)}{10} + \frac{P(1970) - P(1960)}{10} \right\} = \frac{P(1980) - P(1960)}{20}.$$

The same holds for every other time except for the two end points. For them, it is hard to do better than approximating $P'(1950)$ as

$$\frac{P(1960) - P(1950)}{10}$$

and the same thing for $P'(2000)$.

The resulting numbers are in Table 1.

(c) There really is not a much better way to plot this than just connecting up the dots.

(d) Measuring $P(t)$ more frequently and accurately would be one approach. You might also try smoothing out the graph of P (which is rather jumpy) by fitting a curve to the data, and then using the derivative of this curve for P' . Neither of these approaches seem great; getting the derivative accurately is often an issue.

41. It looks like (a) is f , (b) is f' , and (c) is f'' , more or less because no other choice makes sense. With this choice, increasing functions go with positive derivatives and decreasing functions go with negative derivatives.

SECTION 2.8

4. (a) The function f is increasing where f' is positive, that is, on the intervals $(-2, -1)$ and $(0, 1)$. The function f is decreasing where f' is negative, that is, on the intervals $(-1, 0)$ and $(1, 2)$.

(b) The function f will have a local maximum where f' switches from positive to negative, that is, at $x = -1$ and at $x = 1$. The function f will have a local minimum where f' switches from negative to positive, that is, at $x = 0$. There

will also be local minima at the endpoints $x = \pm 2$ of the interval on which f is defined.

(c) I'll let you make the sketch.

5. The function f' is increasing where its derivative, f'' , is positive, which happens where f is concave up. The function f appears to be concave up on the interval $(2, 5)$.

The function f' is decreasing where its derivative, f'' , is negative, which happens where f is concave down. This appears to be on the intervals $(-2, 2)$ and $(5, 9)$.

9. Let $D(t)$ be the amount of the national deficit as a function of time. If we say that the deficit is increasing, this means that $D'(t)$ is positive; if we say that the rate of increase is decreasing, this means that $D''(t)$ is negative. Note that the function $D(t)$ is therefore increasing and concave down. (For a picture of a function such as this, consider the function $f(x)$ from problem 5, on the interval $(5, 6.5)$.)

12. (a) The particle is moving to the left when its derivative is negative, i.e., between $t = 2$ and $t = 4$. It is moving to the right at other times.

(b) The acceleration is the second derivative of the position; so the acceleration is positive when the function is concave up, i.e., for $t > 3$. The acceleration is negative when $t < 3$.

14. Let $d(t)$ be the depth of the coffee as a function of time. Certainly $d(t)$ is always increasing, since we're pouring coffee into the cup, the depth is increasing over time. So $d'(t)$ should be positive everywhere. At first, while the mug's radius decreases, the depth should increase at an increasing rate: $d''(t)$ should be positive. (You can picture this by remembering that if the pouring rate is constant over time, the depth will rise more quickly as the cup narrows.) Later, after the cup has filled past the narrowest point in the cup, the depth will increase at a decreasing rate: $d''(t)$ is now negative. The inflection point corresponds to the time when the coffee line passes the narrowest part of the cup. An approximate sketch is shown in Figure 5

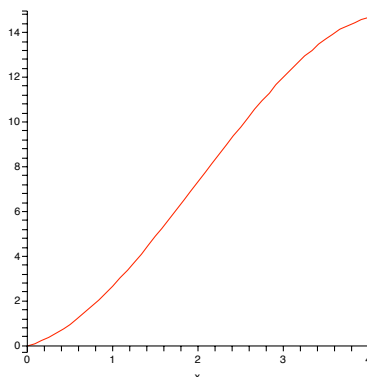


FIGURE 5. Depth of coffee as function of time

16. (a) f is increasing on intervals where $f'(x) \geq 0$, i.e., on $(1, 6)$ and on $(8, 9)$. It is decreasing on $(0, 1)$ and on $(6, 8)$, where $f'(x) \leq 0$.
- (b) There is therefore a local maximum at $x = 6$, and there are local minima at $x = 1$ and at $x = 8$. If $[0, 9]$ is the entire domain of f , then some people would say that f has local maxima at $x = 0$ and at $x = 9$, on the grounds that at no other nearby point in the domain of f does f have larger values than it does at 0 and at 9.
- (c) f is concave up on the intervals $(0, 2)$, $(3, 5)$ and $(7, 9)$, where f' is increasing; it is concave down on the intervals $(2, 3)$ and $(5, 7)$ where f' is decreasing.
- (d) There are inflection points at $x = 2$, $x = 3$, $x = 5$, and $x = 7$.
- (e) Figure 6 is a rough sketch. It's hard to get the heights right and to show the right curvature both.

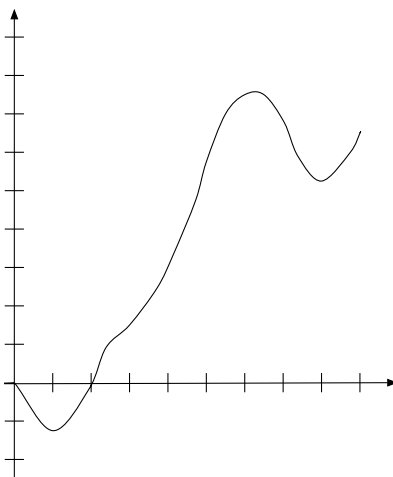


FIGURE 6. Problem 2.8.16: $y = f(x)$.

17. You want a function that is decreasing and concave down. There's one possibility in Figure 7.
22. A function satisfying the conditions listed is shown in Figure 8.
26. What I can say is that $f(x) = (\sqrt{\pi}/2)\operatorname{erf}x + c$, where erf is the error function. But what they want you to do is to observe that e^{-x^2} is always positive, that it is increasing for $x < 0$, and that it is decreasing for $x > 0$ (graph it if these facts are not obvious). Armed with these observations, you can now say that f is increasing everywhere, that it is concave up for $x < 0$, and that it is concave down for $x > 0$.

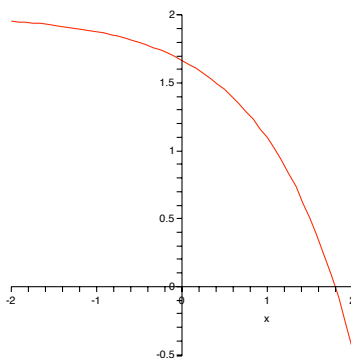


FIGURE 7. Problem 2.8.17: a decreasing, concave down function.

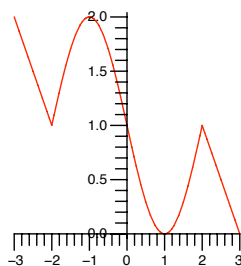


FIGURE 8. Problem 2.8.22: A function satisfying various conditions

28. (a) The derivative is

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{[(x+h)^4 - 2(x+h)^2] - [x^4 - 2x^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2x^2 - 4xh - 2h^2] - [x^4 - 2x^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 4xh - 2h^2}{h} \\
 &= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3 - 4x - 2h) \\
 &= 4x^3 - 4x.
 \end{aligned}$$

The second derivative is therefore

$$\begin{aligned}
 f''(x) &= \lim_{h \rightarrow 0} \frac{[4(x+h)^3 - 4(x+h)] - [4x^3 - 4x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[4x^3 + 12x^2h + 12xh^2 + 4h^3 - 4x - 4h] - [4x^3 - 4x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12x^2h + 12xh^2 + 4h^3 - 4h}{h} \\
 &= \lim_{h \rightarrow 0} (12x^2 + 12xh + 4h^2 - 4) \\
 &= 12x^2 - 4.
 \end{aligned}$$

- (b) The derivative $f'(x) = 4x(x-1)(x+1)$ is positive for $x > 1$ and for $-1 < x < 0$. It is negative for $x < -1$ and for $0 < x < 1$. It follows that $f(x)$ is increasing for $x > 1$ and for $-1 < x < 0$, and that $f(x)$ is decreasing for $x < -1$ and for $0 < x < 1$.
- (c) The second derivative $f''(x) = 12(x - \sqrt{3}/3)(x + \sqrt{3}/3)$ is positive for $x > \sqrt{3}/3$ and for $x < -\sqrt{3}/3$. It is negative for $-\sqrt{3}/3 < x < \sqrt{3}/3$. It follows that $f(x)$ is concave up for $x > \sqrt{3}/3$ and for $x < -\sqrt{3}/3$, and that $f(x)$ is concave down for $-\sqrt{3}/3 < x < \sqrt{3}/3$.
- 30.** If $g(x)$ is an antiderivative for $f(x)$, so that $g'(x) = f(x)$, then g is increasing to the left of the root for f , and decreasing to the right of this root. Only function (a) satisfies these constraints. It looks like g should be concave down more or less everywhere, except that it may be very weakly concave up near the right of the picture if f is increasing there—it's hard to tell from the sketch. In any case, function (a) again satisfies these constraints.
- 31.** The graph of F should start at $(0, 1)$, decrease until it hits a minimum at $x = 2$, increase again, making up perhaps $1/3$ of the height lost, until it reaches a maximum at $x = 3$, then drop down to have a horizontal asymptote at about the height of the minimum at $x = 2$. In the absence of a vertical scale on the graph in the problem, it's hard to say anything about the extent of the vertical changes in the graph of F .
- 32.** Figure 9 shows the original curve (dashed) and its antiderivative (solid line). In your plot, the relative heights of these curves may be different from mine, depending on the scale of the axes.

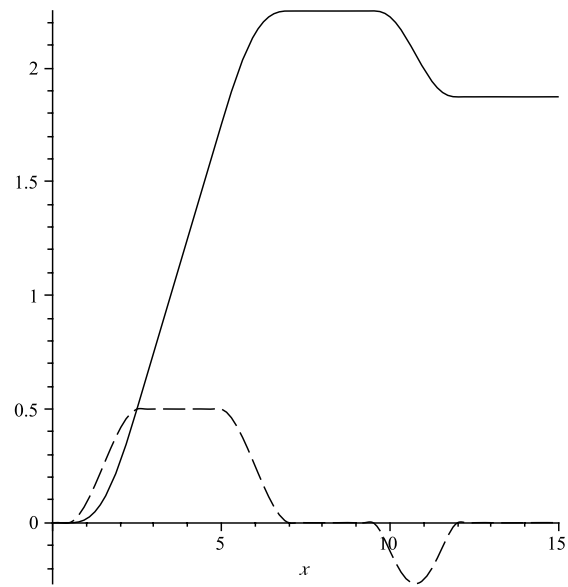


FIGURE 9. Problem 2.8.32: A function (dashed) and its antiderivative (solid).