CALCULUS A
HOMEWORK 7 SOLUTIONS

SECTION 3.4

4. $f(x) = \sin x$ and $g(x) = \cot x$; so $f'(x) = \cos x$ and $g'(x) = -\csc^2 x$. Thus

$$\frac{dy}{dx} = f'(g(x))\, g'(x) = -\cos(\cot x)\ \csc^2 x.$$

8. $F'(x) = 100(4x - x^2)^{99}(4 - 2x)$.

12. $f(t) = (1 + \tan t)^{1/3}$; so $f'(t) = \frac{1}{3}(1 + \tan t)^{-2/3} \sec^2 t$.

17. $y' = 1\, e^{-kx} - kx\, e^{-kx}$.

18. $y' = -2e^{-2t} \cos(4t) - 4e^{-2t} \sin(4t)$.

22. $y' = 10^{1-x^2}(\ln 10)(-2x)$.

23. $y' = 3\left(\dfrac{x^2+1}{x^2-1}\right)^2 \cdot \dfrac{(x^2-1)\,2x - (x^2+1)\,2x}{(x^2-1)^2} = 3\left(\dfrac{x^2+1}{x^2-1}\right)^2 \cdot \dfrac{-4x}{(x^2-1)^2}$.

27. First use the Quotient Rule, then the Chain Rule. You get

$$\begin{aligned} y' &= \frac{\sqrt{r^2+1}\cdot 1 - r\frac{d}{dr}\sqrt{r^2+1}}{r^2+1} \\ &= \frac{\sqrt{r^2+1}\cdot 1 - r\frac{2r}{2\sqrt{r^2+1}}}{r^2+1} \end{aligned}$$

Like many of these answers, this can probably be cleaned up some, but as before, I'm not interested in watching you do this algebra.

34. If you can do this one, then you can differentiate practically anything:

$$\begin{aligned} y' &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \frac{d}{dx}\left(x + \sqrt{x+\sqrt{x}}\right) \\ &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1 + \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{d}{dx}\left(x+\sqrt{x}\right)\right) \\ &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot \left(1 + \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \left(1 + \frac{1}{2\sqrt{x}}\right)\right). \end{aligned}$$

35. $y' = -\left(\sin\sqrt{\sin(\tan(\pi x))}\right) \cdot \dfrac{1}{2\sqrt{\sin(\tan(\pi x))}} \cdot \cos(\tan(\pi x)) \cdot \sec^2(\pi x) \cdot \pi$.

46. At a general point x, the derivative will be

$$y' = \frac{\left(\sqrt{2-x^2}\right)\frac{d}{dx}|x| - |x| \cdot \frac{-2x}{2\sqrt{2-x^2}}}{2-x^2}.$$

At $x = 1$, $(d/dx)|x| = 1$ (the slope of $y = |x|$ at $x = 1$; and

$$y' = \frac{\left(\sqrt{2-1^2}\right) \cdot 1 - |1| \cdot \frac{-2}{2\sqrt{2-1^2}}}{2-1^2} = \frac{1+1}{2-1^2} = 2.$$

The tangent line is therefore the line of slope 2 through the point $(1,1)$, i.e., $y-1 = 2(x-1)$, or $y = 2x-1$. The curve and its tangent line are shown in Figure 1.

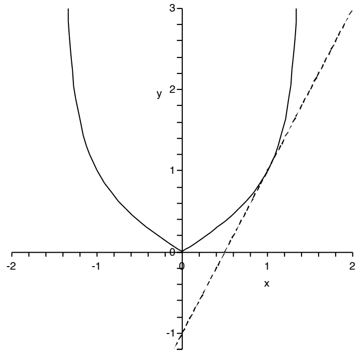

FIGURE 1. Prob. 3.4.46: The bullet-nose curve and its tangent line.

49. The derivative is $f'(x) = 2\cos x + 2\sin x \cos x = 2(\cos x)(1+\sin x)$. This is 0 whenever $\cos x = 0$, that is, at odd multiples of $\pi/2$. It is also 0 when $\sin x = -1$, but these points are all included among those where $\cos x = 0$. The horizontal tangents are thus at $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$.

53. **(a)** $h'(1) = f'(g(1))g'(1) = f'(2)g'(1) = 5 \cdot 6 = 30$.
(b) $H'(1) = g'(f(1))f'(1) = g'(3)f'(1) = 9 \cdot 4 = 36$.

56. Reading the slopes off these graphs is not trivial, so your approximations may differ from mine; but I think that roughly
(a) $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) = (-1)(-1) = 1$.
(b) $g'(x) = f'(x^2) \cdot 2x$, so $g'(2) = 4f'(4) = 4(2) = 8$.

57. $h'(0.5) = f'(g(0.5))g'(0.5) = f'(0.1)g'(0.5)$. My best guess estimate for $g'(0.5)$ is that it's the average of the slopes of the line between $(0.5, g(0.5))$ and $(0.6, g(0.6))$ and the line between $(0.5, g(0.5))$ and $(0.4, g(0.4))$. This average, we have noted before, can be computed rather easily: it is

$$\frac{1}{2}\left[\frac{g(0.6)-g(0.5)}{0.1} + \frac{g(0.5)-g(0.4)}{0.1}\right] = \frac{g(0.6)-g(0.4)}{0.2},$$

the slope of a secant line straddling the point $(0.5, g(0.5))$. In this case, this slope is

$$g'(0.5) \doteq \frac{0.05 - 0.17}{0.2} = -0.6.$$

By the same reasoning, $f'(0.1) \doteq (18.4 - 12.6)/0.2 = 29$; so that $h'(0.5) \doteq -0.6 \cdot 29 = -17.4$.

62. Just differentiate twice, trying to stay calm as you do so:

$$\begin{aligned} f(x) &= xg(x^2) \\ f'(x) &= 1g(x^2) + xg'(x^2)2x = g(x^2) + 2x^2g'(x^2) \\ f''(x) &= g'(x^2)2x + 4xg'(x^2) + 2x^2g''(x^2)2x = 4x^3g''(x^2) + 6xg'(x^2). \end{aligned}$$

66. This problem is the beginning of a pretty important technique in the study of differential equations, turning differential equations into associated polynomial equations. You saw it here first!

In any case, if $y = e^{rx}$, then $y'' - 4y' + y = r^2e^{rx} - 4re^{rx} + e^{rx} = (r^2 - 4r + 1)e^{rx}$. The exponential function is always positive, so $y'' - 4y' + y = 0$ if and only if $r^2 - 4r + 1 = 0$, i.e., if and only if

$$r = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}.$$

71. **(a)** $B'(t) = 0.35\cos(2\pi t/5.4)(2\pi/5.4) = 0.1296\pi\cos(2\pi t/5.4)$.
(b) $B'(1) = 0.1296\pi\cos(2\pi/5.4) = 0.1613$.

74. **(a)** As $t \to +\infty$, $e^{-kt} \to 0$; so $\lim_{t\to+\infty} p(t) = 1/(1 + a \cdot 0) = 1$.
(b) The rate of spread is the derivative

$$p'(t) = -\frac{-ake^{-kt}}{(1 + ae^{-kt})^2} = \frac{ake^{-kt}}{(1 + ae^{-kt})^2}.$$

(c) The plot is in Figure 2. It appears as though $p(t) = 0.8$ near $t = 7.4$ hours. A more accurate estimate using *Maple's* **fsolve** command is $t \doteq 7.3777589$.

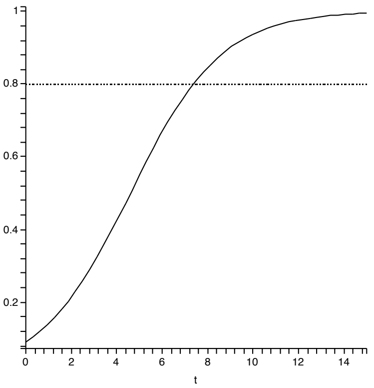

FIGURE 2. Prob. 3.4.74: 80% saturation in rumor spreading.

76. **(a)** dV/dr represents the rate of change of V as a function of r, while dV/dt is the rate of change of V as a function of t. I'm not sure what else they want one to say, but you can think about how you would approximate each, and what the units would be. Since $V = \frac{4}{3}\pi r^3$, we can get dV/dr just from geometry: $dV/dr = 4\pi r^2$. Think about why this is the surface area of the balloon.

(b) The Chain Rule says that

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

93. $y = f(g(x))$, so we are really just being asked for the second derivative of $f(g(x))$, which we can get by using the Chain Rule and the Product Rule. It's simpler in Newton's notation than in Leibniz', I think:

$$\begin{aligned}\frac{d}{dx}(f(g(x))) &= f'(g(x))g'(x)\\ \frac{d^2}{dx^2}(f(g(x))) &= f'(g(x))g''(x) + f''(g(x))g'(x)g'(x).\end{aligned}$$

In Leibniz' notation, this looks like

$$\frac{dy}{du}\frac{d^2u}{dx^2} + \frac{d^2y}{dx^2}\left(\frac{du}{dx}\right)^2,$$

as claimed.

Of course, you could do the whole calculation in Leibniz' notation if you prefer.

Section 3.5

2. **(a)** $-\sin x + \frac{1}{2\sqrt{y}} \cdot y' = 0$; so $y' = 2(\sin x)\sqrt{y}$.

(b) $y = (5 - \cos x)^2$; so $y' = 2(5 - \cos x)(\sin x) = 2(\sin x)\sqrt{y}$.

(c) I just did that.

5.

$$\begin{aligned}x^2 + xy - y^2 &= 4\\ 2x + y + xy' - 2yy' &= 0\\ (x - 2y)y' &= -2x\\ y' &= \frac{2x}{2y - x}.\end{aligned}$$

8.

$$\begin{aligned}y^5 + x^2y^3 &= 1 + ye^{x^2}\\ 5y^4y' + (2xy^3 + x^2 3y^2 y') &= 0 + (y'e^{x^2} + ye^{x^2}2x)\\ 5y^4y' + 3x^2y^2y' - e^{x^2}y' &= -2xy^3 + 2xye^{x^2}\\ y'(5y^4 + 3x^2y^2 - e^{x^2}) &= -2xy^3 + 2xye^{x^2}\\ y' &= \frac{-2xy^3 + 2xye^{x^2}}{5y^4 + 3x^2y^2 - e^{x^2}}.\end{aligned}$$

24. Implicit differentiation gives

$$\begin{aligned} x^2 + 2xy - y^2 + x &= 2 \\ 2x + 2y + 2xy' - 2yy' + 1 &= 0 \\ (2y - 2x)y' &= 2x + 2y + 1. \end{aligned}$$

At the point $(1, 2)$, this means $2y' = 7$; so $y' = 7/2$. The tangent line is the line with this slope through the point $(1, 2)$, namely $y - 2 = \frac{7}{2}(x - 1)$, or $y = \frac{7}{2}x - \frac{3}{2}$.

27.

$$\begin{aligned} 2(x^2 + y^2)^2 &= 25(x^2 - y^2) \\ 4(x^2 + y^2)(2x + 2yy') &= 25(2x - 2yy') \end{aligned}$$

We're interested in y' at the point $(3, 1)$, so

$$\begin{aligned} 4(9 + 1)(6 + 2y') &= 25(6 - 2y') \\ 240 + 80y' &= 150 - 50y' \\ 130y' &= -90 \\ y' &= \frac{-9}{13} \end{aligned}$$

So the equation of the tangent line at $(3, 1)$ is $y - 1 = \frac{-9}{13}(x - 3)$.

A graph of the lemniscate and its tangent line are shown in Figure 3.

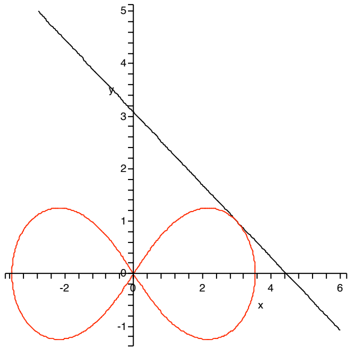

FIGURE 3. Problem 27: $2(x^2 + y^2)^2 = 25(x^2 - y^2$ and its tangent at $(3, 1)$.

37. (a) Sketches at a couple of different scales are shown in Figure 4 and Figure 5. The easy way to do these sketches is just to type the equation $y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$ into the Apple Grapher.

There appear to be 8 points where there are horizontal tangent lines, 4 with $x \approx 0.4$ and 4 with $x \approx 1.6$.

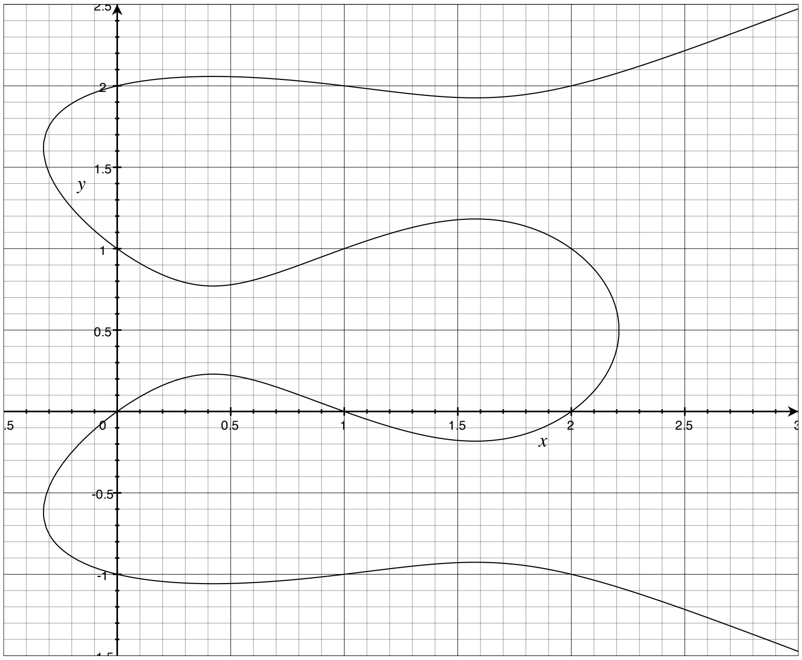

FIGURE 4. Problem 3.5.37: $y(y^2-1)(y-2) = x(x-1)(x-2)$.

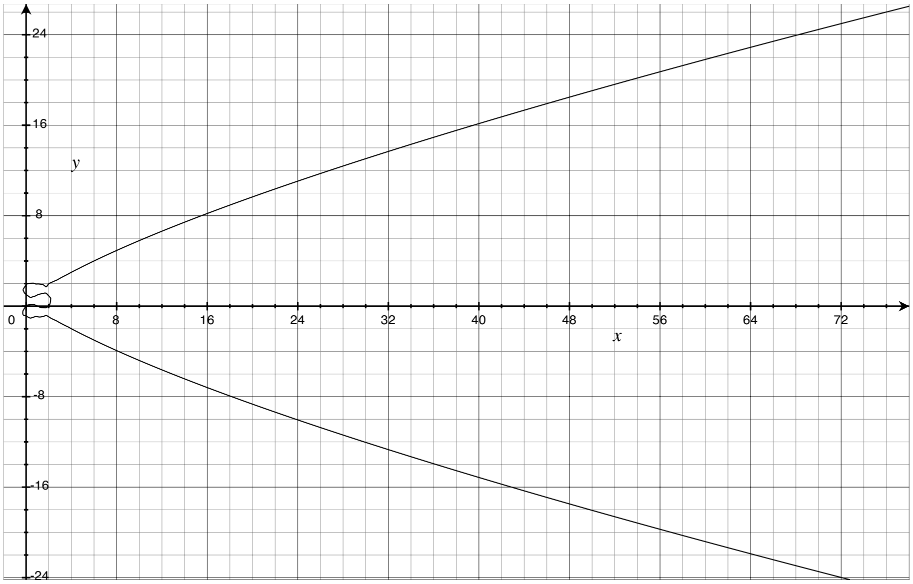

FIGURE 5. Problem 3.5.37: $y(y^2-1)(y-2) = x(x-1)(x-2)$.

(b) Implicit differentiation gives

$$\begin{aligned} y(y^2-1)(y-2) &= x(x-1)(x-2) \\ y^4 - 2y^3 - y^2 + 2y &= x^3 - 3x^2 + 2x \\ (4y^3 - 6y^2 - 2y + 2)\, y' &= 3x^2 - 6x + 2 \\ y' &= \frac{3x^2 - 6x + 2}{4y^3 - 6y^2 - 2y + 2}. \end{aligned}$$

This means that at $(0,1)$, $y' = -1$ and that at $(0,2)$, $y' = \frac{1}{3}$. The tangent lines at these points are therefore $y = -x + 1$ and $y = \frac{1}{3}x + 2$, resp.

(c) The points with horizontal tangent lines occur when $y' = 0$, i.e., when $3x^2 - 6x + 2 = 0$, i.e., at

$$x = \frac{-6 \pm \sqrt{36 - 4(3)(2)}}{6} = 1 \pm \frac{\sqrt{3}}{3}.$$

Roughly this happens at $x = 0.422650$ and $x = 1.577350$, which agrees with our approximations in part (a).

(d) I just thought this might be fun to mess with. The curve $y^2(y^2-1)(y-2) = x(x-1)(x-3)$ is shown in Figure 5.

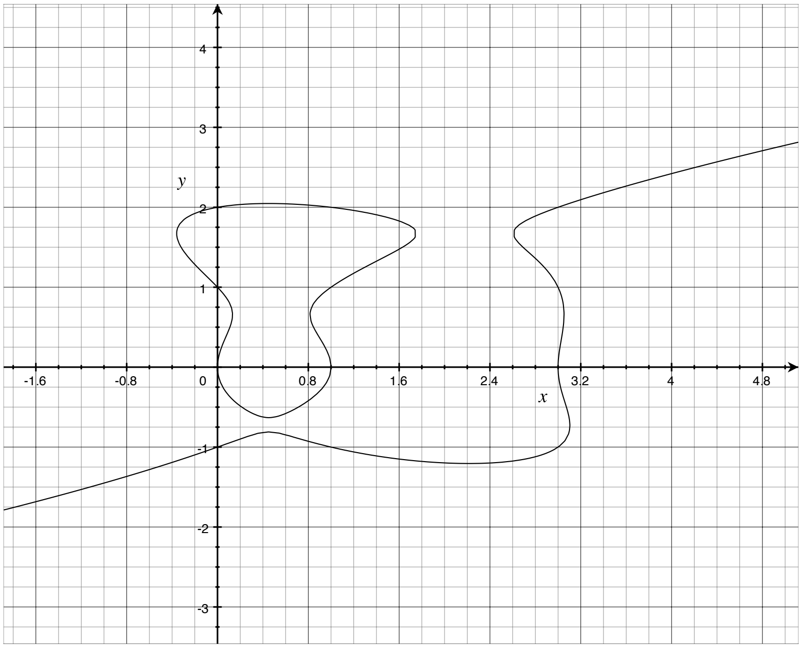

FIGURE 6. Problem 3.5.37: $y^2(y^2-1)(y-2) = x(x-1)(x-3)$.

SECTION 3.6

3. (a) $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/3$; so $\tan(\pi/6) = 1/\sqrt{3}$; so $\tan^{-1}(1/\sqrt{3}) = \pi/6$.

(b) $\cos(\pi/3) = 1/2$; so $\sec(\pi/3) = 2$; so $\sec^{-1}(2) = \pi/3$.

10. Draw the triangle, and you find that $\tan(\sin^{-1}(x)) = \frac{x}{\sqrt{1-x^2}}$.

14. The plot is in Figure 7. The tangent and arctangent are inverse functions, so their graphs are the same curve reflected across the line $y = x$.

18. $y' = \dfrac{2x}{1+x^2}$.

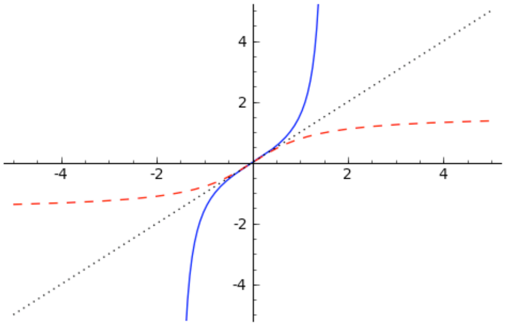

FIGURE 7. Problem 3.6.14:The curves $y = \tan(x)$ (solid), $y = \arctan(x)$ (dashed) and $y = x$ (dotted).

19. $y' = \dfrac{2}{\sqrt{1-(2x+1)^2}}$.

SECTION 3.7

2. $f'(x) = 1\ln x + \dfrac{x}{x} - 1 = \ln x$. Cool. We've got an antiderivative for $\ln x$!

3. $f'(x) = \dfrac{\cos(\ln x)}{x}$.

4. $f'(x) = \dfrac{2(\sin x)(\cos x)}{\sin^2 x} = \dfrac{2\cos x}{\sin x} = 2\cot x$.

5. $f'(x) = -\dfrac{3}{(1-3x)\ln 2}$.

7. $f'(x) = \dfrac{(\ln x)^{-4/5}}{5x}$.

8. The easy way to do this is to observe that $\ln(\sqrt[5]{x}) = \ln(x^{1/5}) = \frac{1}{5}\ln x$, so that $f'(x) = \frac{1}{5x}$.

16. I'd use the same idea:

$$H(z) = \frac{1}{2}\ln\left(\frac{a^2-z^2}{a^2+z^2}\right),$$

so

$$H'(z) = \frac{1}{2}\left(\frac{a^2+z^2}{a^2-z^2}\right)\left(\frac{-2z(a^2+z^2)-2z(a^2-z^2)}{(a^2+z^2)^2}\right) = -\frac{2a^2z}{a^4-z^4}.$$

This is one of those rare instances where simplification (which I do not require you to do) would actually pay off.

17.

$$y' = \frac{-e^{-x}+(1)(e^{-x})+(x)(-e^{-x})}{e^{-x}+xe^{-x}} = \frac{-xe^{-x}}{e^{-x}+xe^{-x}}.$$

22.

$$y' = \frac{x^2 \cdot \frac{1}{x} - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3} = \frac{1}{x^3} - \frac{2\ln x}{x^3}$$
$$y'' = -\frac{3}{x^4} - \frac{2x^3/x - 6x^2\ln x}{x^6} = -\frac{3}{x^4} - \frac{2 - 6\ln x}{x^4} = \frac{-5 + 6\ln x}{x^4}.$$

24. $f'(x) = \left(\frac{1}{\ln\ln x}\right)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right)$. To figure out the domain of f, note that for $\ln\ln\ln x$ to be defined, the outermost ln must have a positive number for its input. So we need $\ln\ln x > 0$, which means that we must have $\ln x > 1$, and so we must have $x > e$.

Section 3.9

6. $f'(x) = \frac{1}{x}$, so $f'(1) = 1$. The line with slope 1 through $(1, \ln 1) = (1, 0)$ is the line $y = x - 1$. So the linearization of $f(x)$ at $a = 1$ is $L(x) = x - 1$.

11. If $f(x) = \sqrt[3]{1-x}$, then $f'(x) = -\frac{1}{3}(1-x)^{-2/3}$, so $f'(0) = -\frac{1}{3}$. The line through $(0, 1)$ with slope $-\frac{1}{3}$ is $y - 1 = -\frac{1}{3}x$, or $y = 1 - \frac{1}{3}x$, as claimed.

The approximation is accurate to within 0.1 whenever $-0.1 \le 1 - \frac{1}{3}x - \sqrt[3]{1-x} \le 0.1$.

Figure 8 shows $L(x) - f(x)$ between $x = -1.3$ and $x = 0.8$, and shows that the error is less than 0.1 roughly for $-1.2 \le x \le 0.8$.

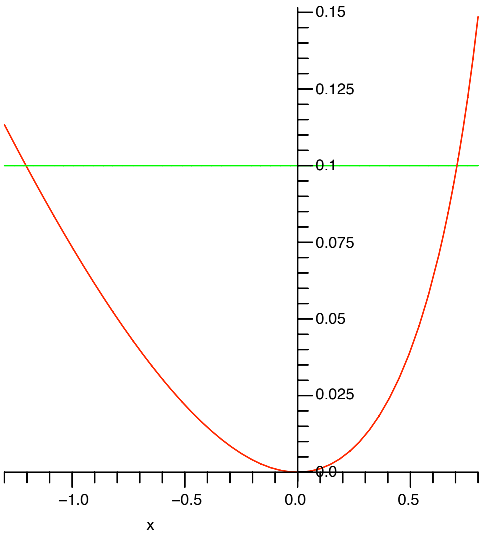

Figure 8. Problem 3.9.11: The error $1 - \frac{1}{3}x - \sqrt[3]{1-x}$.

20. Let $f(x) = (x+1)^6$. Then $f'(x) = 6(x+1)^5$, and $f'(0) = 6$. The tangent line to $f(x)$ at the point $(0, 1)$ is $y - 1 = 6(x - 0)$, so the linear approximation to $f(x)$ near $a = 0$

is $L(x) = 6x+1$. Evaluating $L(x)$ at $x = 0.01$ we have $L(0.01) = 6(0.01)+1 = 1.06$, so $(1.01)^6 = f(0.1) \approx 1.06$.

21. Let $f(x) = \ln(1+x)$. Then $f'(x) = \frac{1}{1+x}$, and $f'(0) = 1$. The tangent line to $f(x)$ at the point $(0, 0)$ is $y = x$, so the linear approximation to $f(x)$ near $a = 0$ is $L(x) = x$. Evaluating $L(x)$ at $x = 0.05$ we have $L(0.05) = 0.05$, so $\ln(1.05) = f(0.5) \approx 0.05$.

36. **(a)** We know that $g(2) = -4$ and that $g'(2) = \sqrt{2^2+5} = 3$. The tangent line to $y = g(x)$ at $x = 2$ is therefore $y + 4 = 3(x - 2)$, or $y = 3x - 10$. This means that

$$\begin{aligned} g(1.95) &\doteq 3(1.95) - 10 = -4.15 \\ g(2.05) &\doteq 3(2.05) - 10 = -3.85. \end{aligned}$$

(b) The second derivative of g is

$$g''(x) = \frac{x}{\sqrt{x^2+5}},$$

which is positive if $x > 0$. The graph of $y = g(x)$ is therefore concave up near $x = 2$. It will therefore lie above its tangent line; so the linear approximations in part (a) will be too low.