CALCULUS A
HOMEWORK 9 SOLUTIONS

SECTION 4.8

2. $\frac{4}{5}x^{10} - \frac{3}{7}x^7 + 3x^4 + c$.

4. At this point, your best bet is to multiply out the original function to get $f(x) = x^3 - 4x^2 + 4x$. The antiderivative is thus $\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2 + c$.

5. $4x^{5/4} - 4x^{7/4} + c$.

6. $x^2 + \frac{3}{2.7}x^{2.7} + c = x^2 + \frac{10}{9}x^{2.7} + c$.

8. The original function is $f(x) = x^{3/4} + x^{4/3}$; so the antiderivative is $\frac{4}{7}x^{7/4} + \frac{3}{7}x^{7/3} + c$.

10. Start by rewriting $g(x)$ as $5x^{-6} - 4x^{-3} + 2$. The antiderivative is then easy:

$$-x^{-5} + 2x^{-2} + 2x + c = \frac{1 - 2x^3 - 2x^6}{x^5} + c.$$

12. $3e^x + 7\tan x + c$.

17. Every antiderivative has the form $F(x) = x^5 - \frac{1}{3}x^6 + c$. The initial condition says that $4 = F(0) = 0^5 - \frac{1}{3}0^6 + c = c$; so we must have $F(x) = x^5 - \frac{1}{3}x^6 + 4$.

32. If $f''(t) = 3/\sqrt{t}$, then $f'(t) = 6\sqrt{t} + c$. The second initial condition says $7 = f'(4) = 6\sqrt{4} + c = 12 + c$; so that $c = -5$ and $f'(t) = 6\sqrt{t} - 5$.

Now we're ready to take the antiderivative again: $f(t) = 4t^{3/2} - 5t + k$. The first initial condition says $20 = f(4) = 4(4)^{3/2} - 5(4) + k = 12 + k$. This means that $k = 8$, and that $f(t) = 4t^{3/2} - 5t + 8$.

35. This time we need to antidifferentiate twice before we can start using the initial conditions. If $f''(x) = 2 + \cos x$, then $f'(x) = 2x + \sin x + c$, and $f(x) = x^2 - \cos x + cx + k$, for some constants c and k. The two initial conditions say that

$$\begin{aligned} 0 &= f(0) = k - 1 \\ 0 &= f(\pi/2) = \frac{c\pi}{2} + k. \end{aligned}$$

Solving those equations gives $k = 1, c = -2/\pi$; so $f(x) = x^2 - \cos x - 2x/\pi + 1$.

37. What the problem tells us in words is that $f'(x) = 2x + 1$ and that $f(1) = 6$. Antidifferentiating gives $f(x) = x^2 + x + c$. The initial condition gives $c = 4$; so $f(x) = x^2 + x + 4$. In particular, $f(2) = 2^2 + 2 + 4 = 10$.

39. The plot is in Figure 1.

40. If $F'(x) = e^x - 2x$ and $F(0) = 1$, then $F(x) = e^x - x^2$. The graphs of F and $f'F'$ are shown in Figure 2.

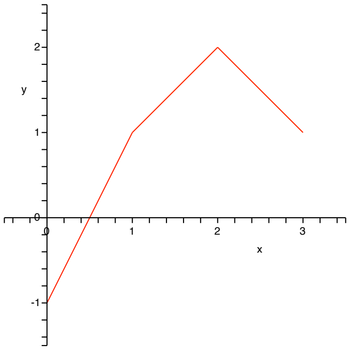

FIGURE 1. Problem 4.9.39: $y = f(x)$.

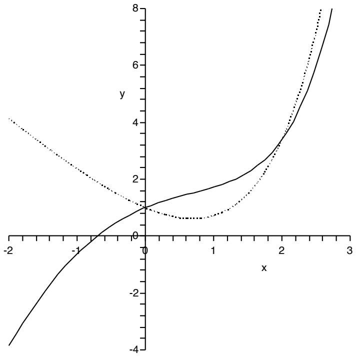

FIGURE 2. Problem 4.9.40: $y = f(x)$ (dotted) and $y = F(x)$ (solid).

42. The velocity is $v(t) = c + 5t + 2t^2 - \frac{2}{3}t^3$. In order to have $v(0) = 3$, it must be the case that $c = 3$ and therefore that $v(t) = 3 + 5t + 2t^2 - \frac{2}{3}t^3$.

Antidifferentiating a second time gives $s(t) = k + 3t + \frac{5}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4$. If $s(0) = 10$, then $k = 10$, and $s(t) = 10 + 3t + \frac{5}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4$.

43. **(a)** The velocity of the stone is $v(t) = -9.8t$ m/s, and the height of the stone above Front Street is $s(t) = -4.9t^2 + 450$ m.

(b) The stone hits the ground when $s(t) = 0$, i.e., at $t = 9.58$ s.

(c) At this time, it is moving at $v(9.58) = -93.9$ m/s.

(d) This time the velocity is $v(t) = -9.8t - 5$ and the position is $s(t) = -4.9t^2 - 5t + 450$. Setting $s(t) = 0$ and solving with the quadratic formula shows that the impact is now after 9.09 s.

49. The velocity of the stone is $v(t) = -32t$ ft/s, and its height is $s(t) = -16t^2 + c$ ft, if c is the height of the cliff. If it hits the ground when $v(t) = -32t = -120$, then it is hitting the ground at $t = 3.75$. At this point, we have

$$0 = s(3.75) = -16(3.75^2) + c = c - 225.$$

The height of the cliff is therefore $c = 225$ feet.

50. The acceleration of the car is -22 ft/s, and its initial velocity is 50 mi/h $= \frac{220}{3}$ ft/s. (One mile equals 5280 feet, just for convenience.) Its velocity is therefore $v(t) = \frac{220}{3} - 22t$. If the brakes are first applied at $s = 0$, then the position is $s(t) = \frac{220}{3}t - 11t^2$. The velocity will be 0 when $t = \frac{10}{3}$; the distance traveled to this point will be $s(\frac{10}{3}) = \frac{1100}{9} \doteq 122.2$ feet.

SECTION 5.1

2. My estimates would be

$$\begin{aligned} L_6 &= (9 + 8.8 + 8.2 + 7.3 + 6 + 4)2 = 86.6 \\ R_6 &= (8.8 + 8.2 + 7.3 + 6 + 4 + 1)2 = 70.6 \\ M_6 &= (8.9 + 8.6 + 7.8 + 6.6 + 5 + 2.9)2 = 79.6. \end{aligned}$$

It's clear that L_6 is an overestimate, since the left-hand rectangles all lie outside the area we are approximating. Similarly, it is clear that R_6 is an underestimate. The average of L_6 and R_6, $T_6 = 78.6$, will be the area of trapezoids that lie below the curve; so it will be an underestimate.

It's not so easy to see that M_6 will be an overestimate. The approach we used in class is sketched in Figure 3. This shows a curve that is concave down, like the one in this problem. A single slice of the area is shown. The area under the midpoint rectangle is equal to the area under the tangent line. Since the tangent line lies above the curve, this area is greater than the area under the curve. The midpoint estimate M_6 is therefore an overestimate.

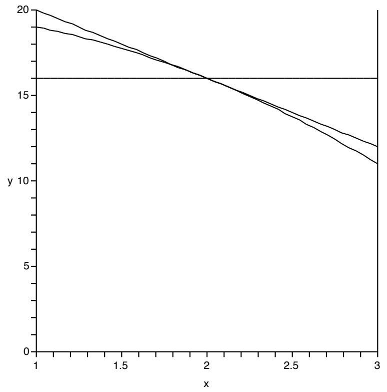

FIGURE 3. Midpoints overestimate concave down functions.

Our Simpson's Rule approach from class would have us take a weighted average of the trapezoid and midpoint sums,

$$S_6 = \frac{2M_6 + T_6}{3} \doteq 79.3.$$

This would be my own best estimate for the area under the curve. Among the three choices given in the problem, it is clear that the midpoint rectangles follow the curve better than the left or right rectangles, so that M_6 is a better estimate than L_6 or R_6.

3. The width of the rectangles is $\pi/8$; so the sum of the areas using right endpoints is

$$\cos\left(\tfrac{\pi}{8}\right) \cdot \tfrac{\pi}{8} + \cos\left(\tfrac{\pi}{4}\right) \cdot \tfrac{\pi}{8} + \cos\left(\tfrac{3\pi}{8}\right) \cdot \tfrac{\pi}{8} + \cos\left(\tfrac{\pi}{2}\right) \cdot \tfrac{\pi}{8} \doteq 0.79077.$$

As the plot in Figure 4 shows, this sum is an underestimate of the actual area.

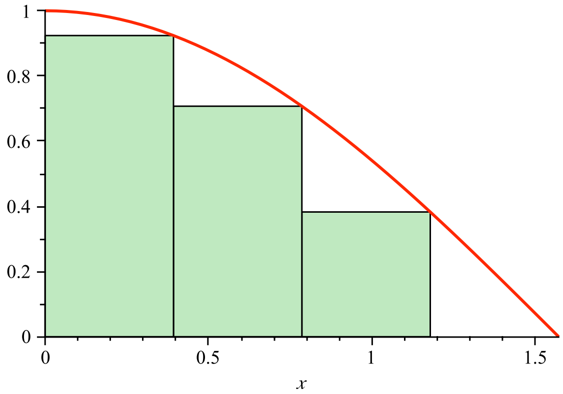

FIGURE 4. Right rectangle approximation to the area under $y = \cos x$.

Similarly, the sum of the rectangles using left endpoints is

$$\cos(0) \cdot \tfrac{\pi}{8} + \cos\left(\tfrac{\pi}{8}\right) \cdot \tfrac{\pi}{8} + \cos\left(\tfrac{\pi}{4}\right) \cdot \tfrac{\pi}{8} + \cos\left(\tfrac{3\pi}{8}\right) \cdot \tfrac{\pi}{8} \doteq 1.18347.$$

The plot in Figure 5 shows that this sum is an overestimate of the actual area.

11. A lower estimate would be to use the left-hand sums and get

$$L_6 = (0 + 6.2 + 10.8 + 14.9 + 18.1 + 19.4)0.5 = 34.7 \text{ ft}.$$

An upper estimate would be gotten by using right hand sums to get

$$R_6 = (6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2)0.5 = 44.8 \text{ ft}.$$

There are various ways you could refine either estimate.

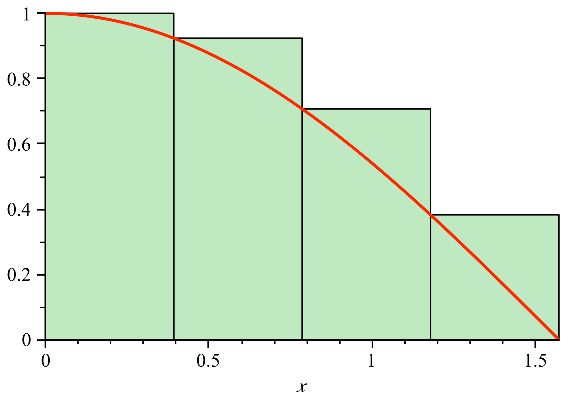

FIGURE 5. Left rectangle approximation to the area under $y = \cos x$.

18. The area will be

$$\lim_{n\to\infty}\left\{\left[\left(4+\tfrac{3}{n}\right)^2+\sqrt{9+\tfrac{6}{n}}\right]\tfrac{3}{n}+\left[\left(4+\tfrac{6}{n}\right)^2+\sqrt{9+\tfrac{12}{n}}\right]\tfrac{3}{n}\right\}+\cdots+\left[\left(4+\tfrac{3n}{n}\right)^2+\sqrt{9+\tfrac{6n}{n}}\right]\tfrac{3}{n}.$$

This looks nicer in Sigma-notation:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[\left(4+\frac{3k}{n}\right)^2+\sqrt{9+\frac{6k}{n}}\right]\frac{3}{n}.$$

22. (a) The area will be

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\cdot\frac{1}{n}$$

(b) The limit will be

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\cdot\frac{1}{n} &= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}i^3\\ &= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2\\ &= \lim_{n\to\infty}\frac{(n+1)^2}{4n^2}\\ &= \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2}\\ &= \lim_{n\to\infty}\frac{1+2/2n+1/n^2}{4}=\frac{1}{4}.\end{aligned}$$

SECTION 5.2

17. This limit is

$$\int_2^6 x\ln(1+x^2)\,dx.$$

21. The integral will be

$$\begin{aligned}\int_{-1}^5 (1+3x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^n (1+3(-1+6\tfrac{i}{n}))\tfrac{6}{n}\\ &= \lim_{n\to\infty}\sum_{i=1}^n \left[-\tfrac{12}{n}+\tfrac{108i}{n^2}\right]\\ &= \lim_{n\to\infty}\left[-12+\tfrac{108}{n^2}\sum_{i=1}^n i\right]\\ &= \lim_{n\to\infty}\left[-12+\tfrac{108}{n^2}\cdot\tfrac{n(n+1)}{2}\right]\\ &= \lim_{n\to\infty}\left[-12+\tfrac{54n+54}{n}\right]\\ &= \lim_{n\to\infty}\left[42+\tfrac{54}{n}\right]=42.\end{aligned}$$

26. The approximation will be

$$\int_0^4 (x^2-3x)\,dx \doteq \sum_{i=1}^8 \left(\left(\frac{i}{2}\right)^2 - 3\left(\frac{i}{2}\right)\right)\frac{1}{2}.$$

The picture of this is in Figure 6.

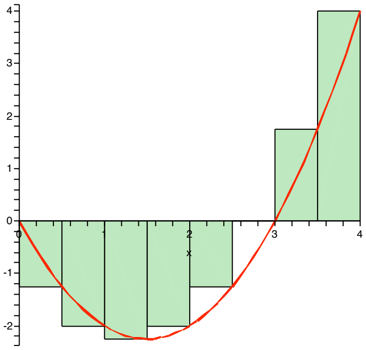

FIGURE 6. Prob. 5.2.26: An 8 slice Riemann sum.

To evaluate the integral exactly, we would need to compute

$$\begin{aligned}
\int_0^4 (x^2 - 3x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{4i}{n}\right)^2 - 3\left(\frac{4i}{n}\right)\right) \frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{64i^2}{n^3} - \frac{48i}{n^2} \right] \\
&= \lim_{n\to\infty} \frac{64}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{48}{n^2} \sum_{i=1}^{n} i \\
&= \lim_{n\to\infty} \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \lim_{n\to\infty} \frac{48}{n^2} \cdot \frac{n(n+1)}{2} \\
&= \lim_{n\to\infty} \frac{64n^2 + 96n + 32}{3n^2} - \lim_{n\to\infty} \frac{24n + 24}{n} = \frac{64}{3} - 24 = -\frac{8}{3}.
\end{aligned}$$

The area we have computed here is the net area under the curve in Figure 6, if the area above the x-axis is positive and the area below the x-axis is negative.