CALCULUS A, LAB 5 SOLUTIONS

1. If

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2, \tag{1}$$

then $T_2'(x) = f'(0) + f''(0)x$ and $T_2''(x) = f''(0)$. Plug in $x = 0$, and you get $T_2(0) = f(0)$, $T_2'(0) = f'(0)$, and $T_2''(0) = f''(0)$.

2. For the first function, we have

$$\begin{aligned} f(x) &= x - \cos x \\ f'(x) &= 1 + \sin x \\ f''(x) &= \cos x, \end{aligned}$$

so that

$$\begin{aligned} f(0) &= -1 \\ f'(x) &= 1 \\ f''(x) &= 1. \end{aligned}$$

This means that $T_1(x) = -1 + x$ and $T_2(x) = -1 + x + \frac{1}{2}x^2$.

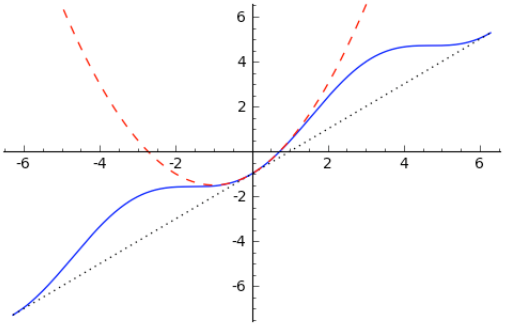

Figure 1. The function $y = x - \cos x$ (solid) and its first (dotted) and second (dashed) degree Maclaurin polynomials.

The function $y = x - \cos x$ and these Maclaurin polynomials are shown in Figure 1. Notice that both $y = T_1(x)$ (dotted) and $y = T_2(x)$ (dashed) are tangent to $y = x - \cos x$ at $x = 0$, but that adding the quadratic term to produce $T_2(x)$ yields a function whose curvature at $x = 0$ matches that of the original function.

For the second function in the problem, we have

$$\begin{aligned} f(x) &= \ln(1+x) \\ f'(x) &= \frac{1}{1+x} \\ f''(x) &= -\frac{1}{(1+x)^2}, \end{aligned}$$

so that

$$\begin{aligned} f(0) &= 0 \\ f'(x) &= 1 \\ f''(x) &= -1. \end{aligned}$$

This means that $T_1(x) = x$ and $T_2(x) = x - \frac{1}{2}x^2$.

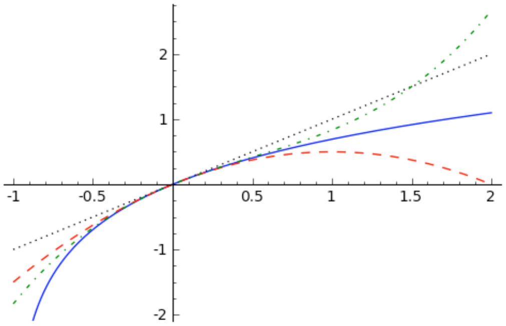

FIGURE 2. The function $y = \ln(1+x)$ (solid) and its first (dotted), second (dashed), and third (dot-dashed) degree Maclaurin polynomials.

The function $y = \ln(1 + x)$ and these Maclaurin polynomials are shown in Figure 2. Again, both $y = T_1(x)$ (dotted) and $y = T_2(x)$ (dashed) are tangent to $y = \ln(1+x)$ at $x = 0$, but adding the quadratic term to produce $T_2(x)$ yields a function whose curvature at $x = 0$ matches that of the original function. For fun, in Figure 2, I've also shown the Maclaurin polynomial of degree 3, $T_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$, in order to show how an x^3 term can be used to push the parabola $y = T_2(x)$ up for positive values of x and down for negative values of x, thereby producing a curve that fits the graph of $y = f(x)$ over a wider interval than does $y = T_2(x)$.

3. The first five derivates of $T_5(x)$ are

$$\begin{aligned}
T_0(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \\
T_0'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 \\
T_0''(x) &= (2\cdot 1)a_2 + (3\cdot 2)a_3x + (4\cdot 3)a_4x^2 + (5\cdot 4)a_5x^3 \\
T_0'''(x) &= (3\cdot 2\cdot 1)a_3 + (4\cdot 3\cdot 2)a_4x + (5\cdot 4\cdot 3)a_5x^2 \\
T_0^{(4)}(x) &= (4\cdot 3\cdot 2\cdot 1)a_4 + (5\cdot 4\cdot 3\cdot 2)a_5x \\
T_0^{(5)}(x) &= (5\cdot 4\cdot 3\cdot 2\cdot 1)a_5.
\end{aligned}$$

This means that at $x = 0$,

$$\begin{aligned}
T_0(0) &= a_0 \\
T_0'(x) &= a_1 \\
T_0''(x) &= (2\cdot 1)a_2 \\
T_0'''(x) &= (3\cdot 2\cdot 1)a_3 \\
T_0^{(4)}(x) &= (4\cdot 3\cdot 2\cdot 1)a_4 \\
T_0^{(5)}(x) &= (5\cdot 4\cdot 3\cdot 2\cdot 1)a_5.
\end{aligned}$$

If the derivatives of f are to agree with the derivatives of the polynomial T_5 at $x = 0$, then we have

$$\begin{aligned}
T_0(0) &= a_0 = f(0) \\
T_0'(x) &= a_1 = f'(0) \\
T_0''(x) &= (2\cdot 1)a_2 = f''(0) \\
T_0'''(x) &= (3\cdot 2\cdot 1)a_3 = f'''(0) \\
T_0^{(4)}(x) &= (4\cdot 3\cdot 2\cdot 1)a_4 = f^{(4)}(0) \\
T_0^{(5)}(x) &= (5\cdot 4\cdot 3\cdot 2\cdot 1)a_5 = f^{(5)}(0).
\end{aligned}$$

Solving these equations for the coefficients $a_0, a_1, a_2, a_3, a_4, a_5$ gives

$$\begin{aligned}
a_0 &= f(0) \\
a_1 &= f'(0) \\
a_2 &= \tfrac{1}{2!}f''(0) \\
a_3 &= \tfrac{1}{3!}f'''(0) \\
a_4 &= \tfrac{1}{4!}f^{(4)}(0) \\
a_5 &= \tfrac{1}{5!}f^{(5)}(0),
\end{aligned}$$

so that

$$T_5(x) = f(0) + f'(0)\,x + \tfrac{1}{2!}f''(0)\,x^2 + \tfrac{1}{3!}f'''(0)\,x^3 + \tfrac{1}{4!}f^{(4)}(0)\,x^4 + \tfrac{1}{5!}f^{(5)}(0)\,x^5.$$

It's obvious that the pattern here will continue, and that the Maclaurin polynomial of degree n will be

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\,x^k.$$

4. Let's just try it with $f(x) = 11x^4 - 7x^3 + 2x^2 - 5x - 9$. Then we have

$$\begin{aligned} f(x) &= 11x^4 - 7x^3 + 2x^2 - 5x - 9 \\ f'(x) &= 4(11)x^3 - 3(7)x^2 + 2(2)x - 5 \\ f''(x) &= 4(3)(11)x^2 - 3(2)(7)x + 2(2) \\ f'''(x) &= 4(3)(2)(11)x - 3(2)(7) \\ f^{(4)}(x) &= 4(3)(2)11, \end{aligned}$$

which means that

$$\begin{aligned} f(0) &= -9 \\ f'(0) &= -5 \\ f''(0) &= 2(2) \\ f'''(0) &= -3(2)(7) \\ f^{(4)}(0) &= 4(3)(2)11. \end{aligned}$$

The Taylor polynomials are therefore

$$\begin{aligned} T_0(x)) &= -9 \\ T_1(x) &= -9 - 5x \\ f''(0) &= -9 - 5x + \frac{2(2)x^2}{2!} = -9 - 5x + 2x^2 \\ f'''(0) &= -9 - 5x + \frac{2(2)x^2}{2!} - \frac{3(2)(7)x^3}{3!} = -9 - 5x + 2x^2 - 7x^3 \\ f^{(4)}(0) &= -9 - 5x + \frac{2(2)x^2}{2!} - \frac{3(2)(7)x^3}{3!} + \frac{4(3)(2)11x^4}{4!} = -9 - 5x + 2x^2 - 7x^3 + 11x^4. \end{aligned}$$

All the higher derivatives are equal to 0, so for every $n > 4$, $T_n(x) = T_4(x)$.

So what can we say about the Taylor polynomials? They are just the original polynomial truncated after the right number of terms! It's pretty easy to see that there isn't anything special about our choice of degree and of coefficients, but that this same thing would work for any other polynomial as well.

5. **(a)** If $f(x) = \sin x$, then the derivatives of f at $x = 0$ follow the pattern $0, 1, 0, -1, 0, 1, 0, -1, \dots$; which means that

$$T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

(b) The derivatives of $f(x) = \cos x$ follow the same pattern starting one position later; so for it,

$$T_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

(c) The derivatives of $f(x) = \frac{1}{x+1} = (x+1)^{-1}$ look like

$$\begin{aligned} f(x) &= (x+1)^{-1} \\ f'(x) &= (-1)(x+1)^{-2} \\ f''(x) &= (-1)(-2)(x+1)^{-3} \\ f'''(x) &= (-1)(-2)(-3)(x+1)^{-4} \\ f^{(4)}(x) &= (-1)(-2)(-3)(-4)(x+1)^{-5}, \end{aligned}$$

and so on. Plugging in $x = 0$ gives $f^{(k)}(0) = (-1)^k k!$, which means that

$$T_n(x) = \sum_{k=0}^{n} \frac{(-1)^k\, k!\, x^k}{k!} = \sum_{k=0}^{n} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots \pm x^n.$$

(d) The function $f(x) = \ln(x+1)$ is an antiderivative of $1/(x+1)$; so its derivatives are the same as the derivatives of the previous function, with $\ln(x+1)$ stuck in at the beginning. Thus, $f(0) = \ln(0+1) = 0$, and for $k > 0$, $f^{(k)}(0) = (-1)^{k-1}(k-1)!$. This means that

$$\begin{aligned} T_n(x) &= \sum_{k=1}^{n} \frac{(-1)^{k-1}\,(k-1)!\, x^k}{k!} = \sum_{k=1}^{n} \frac{(-1)^{k-1} x^k}{k} \\ &= x - \frac{x}{2} + \frac{x^2}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n}. \end{aligned}$$

6. Well, I guess the polynomials we're looking for must then be

$$1 - x^2 + x^4 - x^6 + x^8 - \cdots \pm x^{2n}.$$

7. The graph of $\cos x$ together with $T_2(x)$, $T_4(x)$, $T_6(x)$, and $T_8(x)$ is shown in Figure 3. The polynomials $T_{10}(x)$, $T_{20}(x)$, $T_{30}(x)$, and $T_{40}(x)$ are shown in Figure 4.

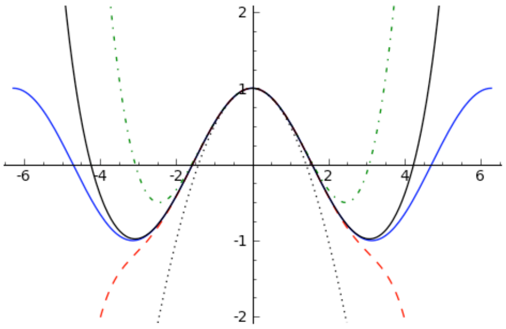

Figure 3. The function $y = \cos x$ and its Maclaurin polynomials $T_2(x)$ (dotted), $T_4(x)$ (dot-dash), $T_6(x)$ (dashed), and $T_8(x)$ (solid).

It appears that as we add more and more terms, the Maclaurin polynomial agrees with the function $y = \cos x$ on a larger and larger interval. It therefore seems that the polynomials eventually get close to f at every point x.

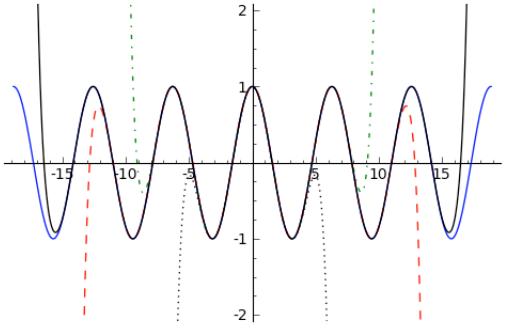

FIGURE 4. The function $y = \cos x$ and its Maclaurin polynomials $T_{10}(x)$ (dotted), $T_{20}(x)$ (dot-dash), $T_{30}(x)$ (dashed), and $T_{40}(x)$ (solid).

I find this amazing because the only thing that went into the Maclaurin series was information about the various derivatives of f at $x = 0$. How on earth can this information encode the behavior of f far away from the origin, many peaks and valleys away fro $x = 0$? Yet that's exactly what happens, and we'll prove it in Calc B. Astounding!

8. Similar plots for $f(x) = (x+1)^{-1}$ are shown in Figure 5 and Figure 6; plots for $f(x) = \ln(x+1)$ are in Figure 7 and Figure 8.

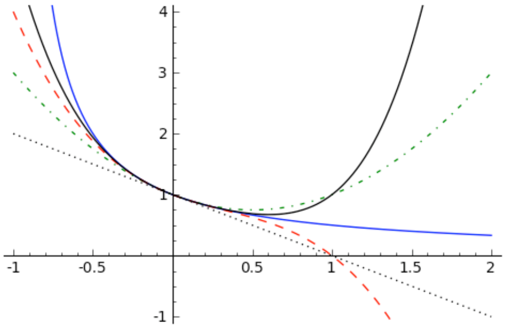

FIGURE 5. The function $y = (x+1)^{-1}$ and its Maclaurin polynomials $T_1(x)$ (dotted), $T_2(x)$ (dot-dash), $T_3(x)$ (dashed), and $T_4(x)$ (solid).

It appears that for both these functions, the Maclaurin polynomials converge roughly for $-1 < x < 1$, and diverge for x outside this interval. We discussed in class that in Calc B we'll see that the interval of convergence is symmetric about the origin. The polynomials probably can't converge past the vertical asymptote at $x = -1$; so they also have to fail to converge past the point $x = 1$. Divergence for $x > 1$ is in some sense an echo of divergence at the vertical asymptote at $x = -1$.

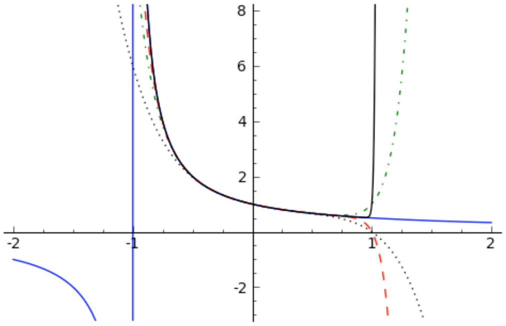

Figure 6. The function $y = (x+1)^{-1}$ and its Maclaurin polynomials $T_5(x)$ (dotted), $T_{10}(x)$ (dot-dash), $T_{15}(x)$ (dashed), and $T_{100}(x)$ (solid).

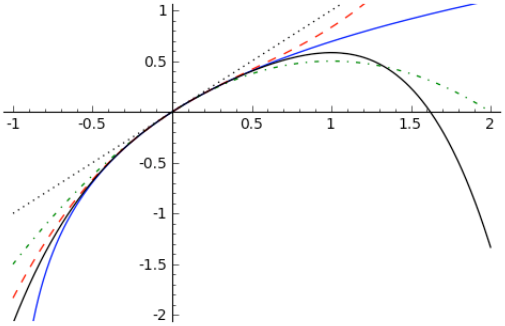

Figure 7. The function $y = \ln(x+1)$ and its Maclaurin polynomials $T_1(x)$ (dotted), $T_2(x)$ (dot-dash), $T_3(x)$ (dashed), and $T_4(x)$ (solid).

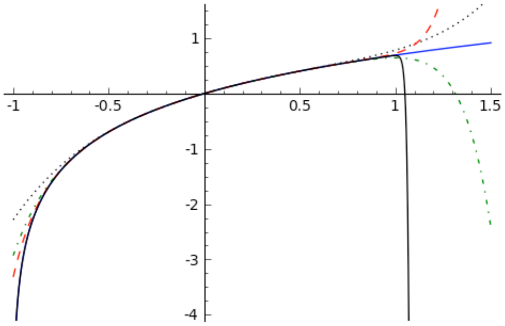

Figure 8. The function $y = \log(x+1)$ and its Maclaurin polynomials $T_5(x)$ (dotted), $T_{10}(x)$ (dot-dash), $T_{15}(x)$ (dashed), and $T_{100}(x)$ (solid).

9. The function $f(x) = 1/(x^2+1)$ and its Maclaurin polynomials are shown in Figures 9 and 10.

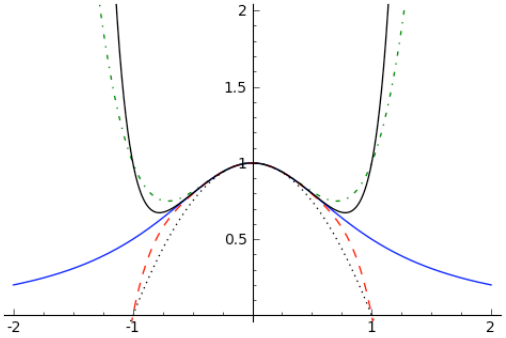

FIGURE 9. The function $y = 1/(x^2+1)$ and its Maclaurin polynomials $T_2(x)$ (dotted), $T_4(x)$ (dot-dash), $T_6(x)$ (dashed), and $T_8(x)$ (solid).

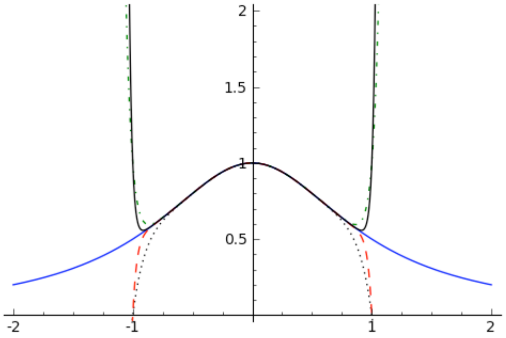

FIGURE 10. The function $y = 1/(x^2+1)$ and its Maclaurin polynomials $T_{10}(x)$ (dotted), $T_{20}(x)$ (dot-dash), $T_{30}(x)$ (dashed), and $T_{40}(x)$ (solid).

Once again, the polynomials seem to converge between 1 and -1, and to diverge outside this interval. This can't be because of any vertical asymptotes on the real line, since $f(x) = 1/(x^2+1)$ has no such asymptotes. What I claimed in our discussion of this in class was that the polynomials converge inside a disk in the complex plane. The function $f(x)$ has vertical asymptotes at $x = \pm i$, so this disk can't have radius greater than 1. The divergence of the sequence of polynomials outside $(-1, 1)$ is a consequence not of vertical asymptotes on the real line, but of these vertical asymptotes at $x = \pm i$.