DISCRETE MATHEMATICS
HOMEWORK 13

1. Argue combinatorially or algebraically or inductively or, well, somehow, that

$$\binom{n+1}{3} = \binom{n}{2} + \binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{3}{2} + \binom{2}{2}.$$

2. A party is planned for 3 married couples, each consisting of one woman and one man. They are to be seated in a row along one side of a long table.

(a) How many seating arrangements total are possible?
(b) In how many seating arrangements do men and women alternate?
(c) In how many seating arrangements is each person adjacent to their spouse?
(d) How many seating arrangements have men and women alternating, and have each person adjacent to their spouse?
(e) How many seating arrangements have men and women alternating, and have no person adjacent to their spouse?

3. A few on binomial coefficents as binomial coefficients:

(a) Expand out $(x+y)^{10}$.
(b) What is the coefficient of x^7 in $(1+x)^{11}$?
(c) What is the coefficient of x^8y^9 in $(3x+2y)^{17}$?
(d) What is the coefficient of x^5y^3 in $(x+y)^{11}$?

4. Show that if n is a positive integer, then

$$\binom{2n}{2} = 2\binom{n}{2} + n^2,$$

(a) By algebraic manipulation of the factorials.
(b) By a combinatorial argument.

5. Show that if S is any finite set, then the number of subsets of S having an even number of elements equals the number of subsets of S having an odd number of elements.

6. Suppose you have a set S of 5 women and 5 men.

(a) How many ways can you select a set of 5 people from S?
(b) How many ways can you select an equal number of women and men from S (either 0, 1, 2, 3, 4, or 5 of each)?
(c) Do you notice anything? Is what you see a coincidence? *Remark:* If you don't notice anything, then make sure you have done the computations in parts (a) and (b) correctly.

7. Suppose, as usual, that we have 4 distinguishable children, Peter, Nicholas, Joanna, and Nina. Suppose further that we have a dog, Jessie, and 7 distinguishable cats, Bagheera, Tycho, Fluffy, Shy, Vex, Squid, and Miss Eliza Tudor. *Remark:* For simplicity, I've left out some poikilotherms. In how many ways can the 8 pets be given to the 4 kids if each child is to get at least 1 pet?

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8. Suppose you have a restaurant with 4 diners who have checked their hats. The hat-check person abruptly decides to quit. As a final statement of the affection in which they view the establishment, they scramble the hats before walking out.

(a) How many ways can the hats be rearranged?

(b) In how many of the rearrangements does no diner end up with the correct hat?

(c) What is the probability of no diner ending up with the right hat?

(d) Can you figure out a way to do this problem if there were more than 4 hats?

(e) What do you suppose would be the probability of no diner getting the correct hat if there were 1000 diners instead of 4? Do you think it is very small (close to 0%), very large (close to 100%) or somewhere in between? Why? Remark: It is OK to guess. It would be nice if the more mathematically adept among you could do this mathematically, but all of you can try to think about it and guess what will happen.