DISCRETE MATHEMATICS
HOMEWORK 5

1. Find all solutions to these Diophantine equations. Convince me that all your solutions work, and that you've got them all.

(a) $29x - 62y = 3$.
(b) $35x - 154y = 7$.

2. In class, we gave a proof of the irrationality of $\sqrt{2}$ that went like this:

Suppose on the contrary that $\sqrt{2}$ is rational. Then we can write $\sqrt{2} = a/b$, where a and b are integers, and $\gcd(a, b) = 1$. A bit of algebra gives

$$2 = \frac{a^2}{b^2}$$
$$2b^2 = a^2. \tag{1}$$

This means that $2 \mid a^2$ (why?), which implies that $2 \mid a$ (why?). We can therefore write $a = 2z$ for some integer z. Plugging this into equation (1) gives

$$2b^2 = (2z)^2$$
$$2b^2 = 4z^2$$
$$b^2 = 2z^2.$$

Now we reason just like we did above: This means that $2 \mid b^2$ (why?), which implies that $2 \mid b$ (why?).

This is a contradiction, though, since we started by assuming that $\gcd(a, b) = 1$, and we have now just shown that a and b have a common divisor of 2. The initial hypothesis must therefore be incorrect, and $\sqrt{2}$ must be irrational.

We then used this proof with some of the 2's changed to 3's to prove the irrationality of $\sqrt{3}$.

(a) The proof above asks "Why" at a couple of points. Come up with reasonable answers.
(b) Can the proof be modified to prove irrationality of $\sqrt[3]{2}$?
(c) The proof cannot be modified to prove the irrationality of $\sqrt{4}$, since $\sqrt{4} = 2$ is rational. Where does the proof break if you try to apply it to $\sqrt{4}$?

3. An arithmetic system has elements a, b, c, and d, with addition and multiplication defined by the following rules:

$+$	a	b	c	d
a	c	d	a	b
b	d	c	b	a
c	a	b	c	d
d	b	a	d	c

$\cdot$	a	b	c	d
a	d	a	c	b
b	a	b	c	d
c	c	c	c	c
d	b	d	c	a

(a) Is there a number in this system you would be tempted to call 0? Why or why not?
(b) Is there a number in this system you would be tempted to call 1? Why or why not?

(c) Which elements x in this system have additive inverses $-x$? What are these inverses?
(d) Which elements x in this system have multiplicative inverses $1/x$? What are these inverses?

Cautionary remark on Problems 3 and 4: Experience from past years shows that on problems like these, students have an almost irresistible obsessive need to conclude that some of the elements a, b, c, d are equal to one another. **This is false. If you think you have proven that these arithmetics have fewer than 4 distinct elements, then you are wrong. You have used some algebraic rule that is false in these arithmetics.** For instance, it's popular for students to say things like, "$x + x = 0$, so $x = 0$." This is true in some arithmetics like the real numbers, but it's false in other arithmetics. In $\mathbb{Z}_6$, for instance, $3 + 3 = 0$, but $3 \neq 0$.

I say this as clearly as I can in class, and still students end up concluding based on incorrect reasoning that $a = b = c = d$ or some such. Really and truly, folks, I mean it: each of the arithmetics in Problems 3 and 4 has four distinct elements, no two equal. Four. Not equal to one another. Clear?

4. Repeat Problem 3 with the arithmetic whose addition and multiplication are defined below:

$+$	a	b	c	d
a	b	c	d	a
b	c	d	a	b
c	d	a	b	c
d	a	b	c	d

$\cdot$	a	b	c	d
a	b	d	b	d
b	d	d	d	d
c	b	d	b	d
d	d	d	d	d

5. Make a table showing the the first through fourth powers of each of the elements of $\mathbb{Z}_5$. For instance, for 2, you should compute

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 3,\ 2^4 = 1.$$

Also make a table of the first through sixth powers of every element of $\mathbb{Z}_7$. Comment on what you observe, and make some conjectures. It would be fine to test your conjectures by looking at other clock arithmetics.

6. In an earlier homework, I asked you to prove or disprove the conjecture that if $a \mid bc$, then $a \mid b$ or $a \mid c$. The conjecture turned out to be false, so we should try to mend it. One possible salvage might be to show that if $a \mid bc$ and if $\gcd(a, b) = 1$, then $a \mid c$. Try to prove or disprove this claim.