DISCRETE MATHEMATICS
HOMEWORK 6

This homework is one of the larger assignments or take-home tests which collectively count for 60% of your grade. Please take it seriously; please don't take it too seriously. You are welcome to use your notes. You are welcome to use computing devices. Please don't use other books, and please don't discuss the test with other living people. Prayer to the saints is, of course, fine, and is encouraged. Please take seriously the comments I wrote on academic integrity at the beginning of the semester.

I've tried to make the test clear and accurate, but nothing in this life comes with a guarantee. Feel free to call me at work or at home (but not after 9PM, please), or to send e-mail to `timm` if you have questions. If I find errors on the test, I will send e-mail to the class; so looking at your mail now and then might be wise.

This test is due in class Monday, October 3. I am tolerant of late homework, but not of late tests.

As is almost always the case on my tests, some of the problems are routine, and some are more challenging. They are in no particular order; easy problems may follow hard problems. Don't panic if there are problems you can't do; nobody in life bats anywhere near 1.000. Do the ones you can do, make a good effort to stretch for the hard ones, stay calm when some of them get past you. But do try a bit of numerical experimentation before you pack any in; focused play is a tool of great power.

Remember also that in mathematics, as in any other discipline, no answer is complete without a justification. A mathematical question can never be answered with "42," or "No," unless there is a calculation or argument to show the correctness of these claims.

Finally, remember that you are writing to be read, and that I'm asking you to convince me that you understand the material. Writing in sentences is always a good idea; your grade school teachers were right about that. Don't give answers like, "As we showed repeatedly in class, ...," even if that's true. I'm interested in whether *you* know how to do the problems; I already know that *I* know how. On the other hand, don't repeat the same proof over and over again on the test. If you already did some proof on Problem 4, then in Problem 2, it is perfectly reasonable to say, "By an identical argument to that in Problem 4, ..."

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Modify our proof of the irrationality of $\sqrt{2}$ to prove the irrationality of $\sqrt[3]{4}$. Mind the details.

2. Find values for the fractions 1/2, 1/3, 1/4, and 1/5 in $\mathbb{Z}_{621}$, or convince me they don't exist.

3. Convince me of the truth or falsehood of the following claims:

(a) If $a \mid b$, then $ac \mid bc$.
(b) If $a \mid c$ and $b \mid c$, then $ab \mid c$.
(c) If $a \mid (b+c)$ and $a \mid (b-c)$, then $a \mid 2b$. Does it also follow that $a \mid b$?

4. Here are a couple on Diophantine equations:

(a) Find solutions in the integers to the equation $79x - 250y = 1$. Convince me that any solutions you find actually work, and that you have found all the solutions.
(b) Do the same thing with the equation $52x - 143y = 1$.
(c) Explain what these problems have to do with the problems of finding 1/79 in $\mathbb{Z}_{250}$ and 1/52 in $\mathbb{Z}_{143}$. Find these fractions if they exist.

5. Compute all possible values of the following quantities. Be sure to explain what you are doing. Work carefully; you may find at least one surprise.

(a) -13 in $\mathbb{Z}_{78}$.
(b) $3/5$ in $\mathbb{Z}_{37}$.
(c) $\sqrt{-1}$ in $\mathbb{Z}_{13}$.
(d) $\sqrt{4}$ in $\mathbb{Z}_{21}$.
(e) $\gcd(1234567890,\ 99)$.
(f) $\gcd(39495952,\ 12472429)$.

6. Explain why Euclid's algorithm computes the greatest common divisor of the two starting numbers. You could do this with a formal proof, or with a careful analysis of a numerical example (patterned on what we did in class). If you do use a numerical example, make it different from the one we used in class.

7. In previous homework, we computed the function $\tau(n)$, the number of positive divisors of n. Suppose we define a new function, $\nu(n)$, to be the number of perfect squares that divide n. For example, the squares that divide 400 are 1, 4, 16, 25, 100, and 400; so $\nu(400) = 6$. Find a formula for $\nu(n)$ in terms of the prime factorization of n,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}.$$

Your approach to doing this will probably be a lot like your approach to the earlier problems of this sort.

You may find it useful in your solution to use the floor function, $\lfloor x \rfloor$, which is defined as x rounded down to the nearest integer. Thus, $\lfloor 4 \rfloor = \lfloor 4.1 \rfloor = \lfloor 4.99 \rfloor = 4$.