

# COUNTEREXAMPLES TO THE 0-1 CONJECTURE

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ABSTRACT. For permutations  $x$  and  $w$ , let  $\mu(x, w)$  be the coefficient of highest possible degree in the Kazhdan-Lusztig polynomial  $P_{x,w}$ . It is well-known that the  $\mu(x, w)$  arise as the edge labels of certain graphs encoding the representations of  $S_n$ . The 0-1 Conjecture states that the  $\mu(x, w) \in \{0, 1\}$ . We present two counterexamples to this conjecture, the first in  $S_{16}$ , for which  $x$  and  $w$  are in the same left cell, and the second in  $S_{10}$ . The proof of the counterexample in  $S_{16}$  relies on computer calculations.

## 1. INTRODUCTION

In studying the representations of Hecke algebras, Kazhdan and Lusztig [7] defined a class of polynomials now known as Kazhdan-Lusztig (KL) polynomials  $P_{x,w}$  that are indexed by pairs of elements in a Coxeter group. These polynomials carry important representation-theoretic and geometric information. Certain coefficients  $\mu(x, w)$  are particularly important representation-theoretically in addition to controlling the recursive structure of the polynomials. While these coefficients  $\mu(x, w)$  are easily seen to take varied (non-negative) values in most Weyl groups, empirical evidence has suggested the following

**0-1 Conjecture.** For  $x, w \in S_n$ ,  $\mu(x, w) \in \{0, 1\}$ .

If this conjecture were true, Kazhdan and Lusztig's construction of the irreducible representations of  $S_n$  (see [7]) would be embodied simply by graphs rather than *edge-labeled* (by the  $\mu(x, w)$ ) graphs. However, as the following theorem shows, this is not the case.

**Theorem 1.** *Identify elements of  $S_{16}$  (resp.,  $S_{10}$ ) with permutations of the set  $\{0, 1, \dots, 9, a, b, c, d, e, f\}$  (resp.,  $\{0, 1, \dots, 9\}$ ). We have the following two equalities:*

1.  $\mu(54109832dc76bafec810d942fa53b6e7) = 5$ .
2.  $\mu(4321098765, 9467182350) = 4$ .

The first case offers the smallest counterexample to the 0-1 Conjecture with both permutations lying in the same left cell. The existence of such an example implies that the graphs describing the irreducible representations do, in fact, need to be edge-labeled. Exhaustive computer calculations by du Cloux [3] and both authors independently have shown that there are no counterexamples in  $S_9$  or below. Hence, the counterexample given in part 2 of Theorem 1 occurs in the smallest possible group. The following corollary is immediate from either part of Theorem 1:

**Corollary 2.** *The 0-1 Conjecture is false.*

So, in this sense, the combinatorics of the symmetric group is not simpler than that of other Weyl groups.

The possibility that  $\mu(x, w) \in \{0, 1\}$  for any  $x, w \in S_n$  was noticed by Lascoux and Schützenberger (see [5]), presumably noticed by Kazhdan and Lusztig, and certainly noticed independently by many others. In fact, Lascoux and Schützenberger [10] showed the 0-1 Conjecture to be true for Grassmannian permutations  $w$ . However, given the difficulty of examining  $S_n$  for  $n \geq 9$  empirically, there has not appeared to be a consensus as to the truth of the conjecture.

Based on the work of Lascoux and Schützenberger, Garsia and McLarnan [5] list three progressively weaker conjectures related to the 0-1 Conjecture. The Lascoux-Schützenberger (L-S) graph has as vertices all members of a left cell. All pairs  $\{x, w\}$  in which  $w$  covers  $x$  in the left weak Bruhat-Chevalley order are edges of this graph, as are the pairs produced by all possible applications of the  $L_i$  of Definition 10. Each such edge  $\{x, w\}$  is easily checked to have  $\mu(x, w) = 1$ . There are three natural questions motivated by this construction:

1. Is this L-S graph identical to the “K-L graph” described by Kazhdan and Lusztig in [7]?
2. If one starts with the L-S graph and follows the recipe of Kazhdan and Lusztig for using the graph to associate a transition matrix to each permutation, does one obtain an irreducible representation for  $S_n$  corresponding to that left cell?
3. If not, then does one at least get some representation of  $S_n$ ?

In Section 5, we elaborate on the computer calculations of the first author from the late 1980s that answered each of these questions in the negative. (Ochiai and Kako [12], in response to hearing about these original computations, confirmed these answers in the pursuit of computing the K-L graphs up through  $S_{15}$ .) Section 2 presents preliminary notation and definitions. Section 3 describes the algorithm used by the first author in 1989 to prove Theorem 1.1. In Section 4, we give the second author’s combinatorial proof of Theorem 1.2.

## 2. PRELIMINARIES

We will consider elements of  $S_n$  as permutations on the set  $\{0, \dots, n-1\}$ . As a generating set  $\mathcal{S}$ , we will take the adjacent transpositions  $s_i = (i, i+1)$  for  $0 \leq i \leq n-2$ . A one-line notation for a permutation  $w$  is afforded by writing the image of the  $n$ -tuple  $[0, \dots, n-1]$  under the action of  $w$ :  $[w(0), w(1), \dots, w(n-1)]$  (we often omit the commas and brackets). The *length function* for  $S_n$  is given by

$$l(w) = |\{0 \leq i < j < n : w(i) > w(j)\}|.$$

In Definition 4, we define the Bruhat-Chevalley partial order on  $S_n$ . (The definition we give is equivalent to more common descriptions such as the tableau criterion; see [1, 4, 6] and the references cited therein.)

**Definition 3.** Let  $x, w \in S_n$ ,  $p, q \in \mathbb{Z}$ . Define the *rank function*  $r_w(p, q) := |\{i \leq p : w(i) \geq q\}|$  and the *difference function*  $d_{x,w}(p, q) := r_w(p, q) - r_x(p, q)$ .

**Definition 4.** We define the *Bruhat-Chevalley partial order* “ $\leq$ ” on  $S_n$  by setting  $x \leq w$  if and only if  $d_{x,w}(p, q) \geq 0$  for all  $p, q$ .

Let  $[x, w] = \{z : x \leq z \leq w\}$  in the Bruhat-Chevalley ordering. Billey and Warrington [2] prove the following result.

**Lemma 5.** *If  $x(i) = w(i)$ ,  $d_{x,w}(i, x(i)) = 0$  and  $z \in [x, w]$ , then  $z(i) = x(i)$ .*

We can view the Bruhat-Chevalley order graphically using “Bruhat pictures” determined by the function  $d_{x,w}$ . A typical picture is shown in Figure 1. Let  $\text{mat}(w)$  refer to the permutation matrix for  $w$ . Entries of  $\text{mat}(x)$  (resp.,  $\text{mat}(w)$ ) are denoted by black disks (resp., open circles). Shading denotes regions in which  $d_{x,w} \geq 1$ . Successively darker shading denotes successively higher values of  $d_{x,w}$ . Positions corresponding to 1’s of both  $\text{mat}(x)$  and  $\text{mat}(w)$  (termed “capitols”) are denoted by a black disk and a larger concentric circle.

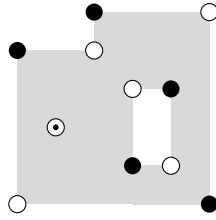


FIGURE 1. Bruhat picture for  $x = [2, 0, 4, 1, 3, 5]$ ,  $w = [5, 2, 3, 1, 4, 0]$ .

While not strictly necessary, these pictures help motivate results such as Lemma 5 and can be very helpful in computing with KL polynomials. In fact, a number of the arguments in Section 3 were arrived at with the aid of these pictures. The reader of that section may benefit from constructing the appropriate Bruhat pictures.

Using the Bruhat-Chevalley order, there are several sets we can associate to any permutation  $w$ . We define the right and left descent sets of  $w$  and the set of flush elements of  $w$  to be

- (1)  $D_R(w) = \{s \in \mathcal{S} : ws < w\}$ ,
- (2)  $D_L(w) = \{s \in \mathcal{S} : sw < w\}$  and
- (3)  $\text{Flush}(w) = \{x \leq w : D_R(x) \supseteq D_R(w) \text{ and } D_L(x) \supseteq D_L(w)\}$ .

We now give a combinatorial definition of the Kazhdan-Lusztig (KL) polynomials applicable to any Coxeter group. For motivation and a more natural definition, we refer the reader to [6, 7]. In order to give the definition succinctly, we set

$$(4) \quad \mu(x, w) = \text{coefficient of } q^{(l(w)-l(x)-1)/2} \text{ in } P_{x,w},$$

and define  $c_s(x) = 1$  if  $xs < x$ ;  $c_s(x) = 0$  if  $xs > x$ .

**Theorem 6** ([7]). *There is a unique set of polynomials  $\{P_{x,w}\}_{x,w \in S_n}$  such that, for all  $x, w \in S_n$ :*

1.  $P_{w,w} = 1$ ,
2.  $P_{x,w} = 0$  when  $x \not\leq w$ ,
3. If  $s \in D_R(w)$ , then

$$(5) \quad P_{x,w} = q^{c_s(x)} P_{x,ws} + q^{1-c_s(x)} P_{xs,ws} - \sum_{\substack{z \leq ws \\ zs < z}} \mu(z, ws) q^{\frac{l(w)-l(z)}{2}} P_{x,z}.$$

The analogous recursion with  $s$  acting on the left holds when  $s \in D_L(w)$ . Further, these polynomials satisfy the degree restriction

$$(6) \quad \deg(P_{x,w}) \leq (l(w) - l(x) - 1)/2 \text{ when } x < w.$$

Note that  $\mu(x, w)$  is the coefficient of the highest possible power of  $q$  in  $P_{x,w}$  and that  $\mu(x, w) = 0$  if  $l(w) - l(x)$  is even.

The complexity of the KL polynomials arises from the sum subtracted off in (5). We now introduce some notation to let us deal with these sums concisely. For  $x, w \in S_n$  and  $s \in D_R(w)$ , let

$$\begin{aligned} \omega_{(\cdot s)}[x, ws] &= \{z : x \leq z < ws, zs < z, l(z) < l(ws) - 1, z \in \text{Flush}(ws)\}, \\ \delta_{(\cdot s)}[x, ws] &= \{z : x \leq z < ws, zs < z, l(z) = l(ws) - 1\}, \\ \theta_{(\cdot s)}[x, ws] &= \delta_{(\cdot s)}[x, ws] \cup \omega_{(\cdot s)}[x, ws], \\ \Theta_{(\cdot s)}[x, ws] &= \sum_{z \in \theta_{(\cdot s)}[x, ws]} \mu(z, ws) q^{\frac{l(w)-l(z)}{2}} P_{x,z}. \end{aligned}$$

Proposition 9.4 will imply that  $\Theta_{(\cdot s)}[x, ws]$  is the sum appearing in (5). Let  $z \in [x, ws]$ . We say that  $z$  is *right  $s$ -flush* for this interval if  $z \in \omega_{(\cdot s)}[x, ws]$ . It is *right  $s$ -coatomic* for this interval if  $z \in \delta_{(\cdot s)}[x, ws]$ . The “left” versions are defined analogously (with “ $(\cdot s)$ ” substituted for “ $(\cdot s)$ ”). We will omit “left” and “right,” as they will be clear from context.

We will need several additional properties of KL polynomials that are not immediately apparent from the definition; we require the following notation:

**Definition 7.** For  $w \in S_n$  and  $0 \leq i_1 < \dots < i_k \leq n - 1$  for  $k \leq n$ , let  $\text{fl}[w(i_1), w(i_2), \dots, w(i_k)]$  be the unique *flattened* permutation  $[v(1), \dots, v(k)] \in S_k$  such that  $v(j) < v(k)$  precisely when  $w(i_j) < w(i_k)$ .

For example, if  $w = 7461098253$ , then

$$\text{fl}[w(0), w(2), w(3), w(5), w(8), w(9)] = \text{fl}[7, 6, 1, 9, 5, 3] = 430521.$$

**Definition 8.** Let

$$\Delta(x, w) = \{i : x(i) \neq w(i) \text{ or } d_{x,w}(i, x(i)) \neq 0\}.$$

If  $\Delta(x, w) = \{d_1, d_2, \dots, d_k\}$  with  $d_i < d_j$  for  $i < j$ , we get two *reduced* permutations by flattening  $x$  and  $w$  with respect to  $\Delta(x, w)$ :

$$\begin{aligned} \tilde{x} &= \text{fl}([x(d_1), x(d_2), \dots, x(d_k)]) \quad \text{and} \\ \tilde{w} &= \text{fl}([w(d_1), w(d_2), \dots, w(d_k)]). \end{aligned}$$

Note that  $\tilde{x}$  and  $\tilde{w}$  are permutations in  $S_k$ . For instance, if

$$(7) \quad x = 6491082753$$

$$(8) \quad w = 9461782350,$$

then  $\Delta(x, w) = \{0, 2, 4, 6, 7, 9\}$ , and

$$(9) \quad \tilde{x} = 350142$$

$$(10) \quad \tilde{w} = 534120.$$

To obtain the Bruhat picture of the pair  $\tilde{x}, \tilde{w}$  from that for  $x, w$ , one simply removes the capitols not surrounded by a shaded region (see Figure 2).

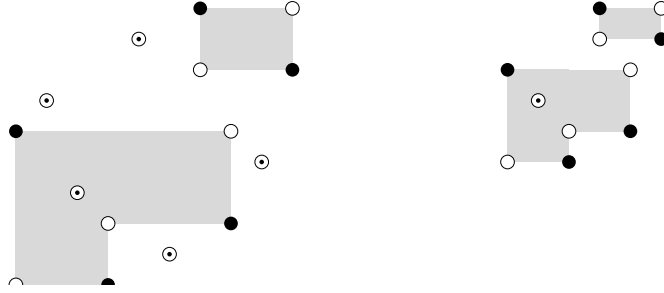


FIGURE 2. Sample Bruhat pictures for (7) and (9).

**Proposition 9.** *The KL polynomials satisfy the following properties:*

1. If  $s \in D_R(w)$ , then  $P_{x,w} = P_{xs,w}$ . If  $s \in D_L(w)$ , then  $P_{x,w} = P_{sx,w}$ .
2.  $P_{x,w} = P_{x^{-1},w^{-1}}$ .
3.  $P_{x,w} = P_{\tilde{x},\tilde{w}}$ .
4. If  $x \notin \text{Flush}(w)$  and  $l(x) < l(w) - 1$ , then  $\mu(x,w) = 0$ .

The first two properties are standard and can be found in [7]. Proof of the third can be found in [2]; the fourth follows from the first property along with (6).

The  $\mu(x,w)$  also satisfy an identity which will be integral to the proof in Section 3. To state it, we make the following definitions:

**Definition 10.** Let  $\mathcal{L}_k$  be the set of permutations  $w$  for which  $s_k w < w$  or  $s_{k+1} w < w$ , but not both. Define an operator  $L_k$  acting on  $\mathcal{L}_k$  by setting

$$(L_k w)^{-1}(j) = \begin{cases} w^{-1}(k+2), & \text{if } j = k, \\ w^{-1}(k), & \text{if } j = k+2, \\ w^{-1}(j), & \text{otherwise.} \end{cases}$$

In other words,  $\mathcal{L}_k$  consists of all permutations in which  $k, k+1, k+2$  do not appear either in increasing or decreasing order; and  $L_k w$  is obtained from  $w$  by interchanging  $k, k+2$ . For instance,  $L_2[3, 1, 4, 0, 2] = [3, 1, 2, 0, 4]$ . The operator  $L_k$  is called an elementary Knuth transformation. It is intimately connected to the Robinson-Schensted correspondence discussed below; for details, see [4, 8, 9].

**Definition 11.** For  $x$  and  $w$  comparable under the Bruhat-Chevalley order, set

$$\mu[x,w] = \begin{cases} \mu(x,w), & \text{if } x \leq w, \\ \mu(w,x), & \text{otherwise.} \end{cases}$$

**Theorem 12** ([7]). *If  $x, w \in \mathcal{L}_k$ , then  $\mu[x,w] = \mu[L_k x, L_k w]$ .*

### 3. COMPUTER PROOF OF THEOREM 1.1

The example of Theorem 1.1 was found via a computer search for a counterexample to the 0-1 Conjecture. As looking at every pair of permutations even in  $S_{10}$  is prohibitively expensive, we will cut down our search space by searching for a counterexample that is minimal in some sense. In particular, we will search for a counterexample  $\{x, w\}$  in  $S_n$  with  $x$  and  $w$  in the same left cell which minimizes in order the following parameters.

1.  $n$ : i.e.,  $\mu(u, v) \leq 1$  for all  $u, v \in S_{n-1}$  in the same left cell.
2.  $l(w) - l(x)$ : i.e.,  $u, v \in S_n$  in the same left cell with  $l(v) - l(u) < l(w) - l(x)$  implies  $\mu(u, v) \leq 1$ .
3.  $l(w)$ : i.e.,  $v \in S_n$  with  $l(v) < l(w)$  implies that there does not exist a  $u \in S_n$  in the same left cell as  $v$  having  $\mu(u, v) > 1$ .

The key to searching efficiently turns out to be the Robinson-Schensted correspondence, which we now recall. The material in this section will be presented briefly—a more detailed exposition can be found in [5] (also see [4, 9]).

Let  $\lambda$  be a partition of  $n$  (denoted  $\lambda \vdash n$ ) with  $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ . We associate a Ferrers diagram consisting of left-justified rows of boxes with  $\lambda_i$  boxes in the  $i$ -th row from the bottom. A *standard tableau of shape*  $\lambda$  is an injective filling of these boxes with  $0, 1, 2, \dots, n-1$  such that the entries increase from left to right on rows and from bottom to top on columns. The *column word*,  $\text{cwd}(T)$ , of a tableau  $T$  is obtained by reading the columns of  $T$  from top to bottom starting with the leftmost column. The *row word*,  $\text{rwd}(T)$ , of  $T$  is obtained by reading the rows of  $T$  from left to right starting with the top row. Any word that can be obtained in this way from some tableau is called a *tableau word*. The *descent set*  $D(T)$  of a tableau  $T$  is the set of indices  $i$  for which  $i+1$  is strictly to the north and weakly to the west of  $i$ . In other words,

$$D(T) = D_L(\text{rwd}(T)) = D_L(\text{cwd}(T)).$$

The Robinson-Schensted correspondence gives a bijection between the elements  $w \in S_n$  and the pairs of tableaux of the same shape  $\lambda \vdash n$ . For the specifics of the bijection, see, e.g., [4]. Via the Bruhat-Chevalley order on permutations, this correspondence induces a partial order on pairs of tableaux of the same shape which we will also denote by “ $\leq$ ”. The *left cell* indexed by the tableau  $Q$  consists of all pairs  $(P_w, Q)$  where  $P_w$  has the same shape as  $Q$ . Below we illustrate these definitions:

$$(11) \quad x = 4265013 \leftrightarrow (P_x, Q_x) = \left( \begin{array}{|c|c|} \hline 4 & 6 \\ \hline 2 & 5 \\ \hline 0 & 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 1 & 3 \\ \hline 0 & 2 & 6 \\ \hline \end{array} \right),$$

$$(12) \quad \text{rwd}(P_x) = 4625013,$$

$$(13) \quad \text{cwd}(P_x) = 4206513.$$

In order to describe the edge-labeled graphs defined by Kazhdan and Lusztig corresponding to *irreducible* representations of  $S_n$ , we need only consider pairs  $x, w$  lying in the same left cell. Thus, a counterexample among such pairs shows that the edge-labeling is necessary.

Fortunately, there is complete redundancy amongst the left cells with respect to the values of the  $\mu[x, w]$ :

**Theorem 13** ([7]). *Let  $P_x, P_w, Q$  and  $Q'$  be tableaux of the same shape. Let  $x, w, x'$  and  $w'$  correspond, under the Robinson-Schensted correspondence, to the following pairs of tableaux:*

$$\begin{aligned} x &\leftrightarrow (P_x, Q), & w &\leftrightarrow (P_w, Q), \\ x' &\leftrightarrow (P_x, Q'), & w' &\leftrightarrow (P_w, Q'). \end{aligned}$$

*Then  $\mu[x, w] = \mu[x', w']$ .*

With Theorem 13, to search for a minimal counterexample, we effectively need only search over pairs  $(P_x, Q), (P_w, Q)$  of tableaux of the same shape without regard to  $Q$ . One can get a sense of the savings by noting that  $|S_{16}| = 20,922,789,888,000$ , while the number of standard Young tableau of size 16 is a mere 46,206,736. The task of considering all *pairs* of permutations is clearly infeasible. While one still needs to look at pairs of tableaux (of the same shape), even naively we need only consider  $|S_{16}|$  pairs. And, with the proper filters, we can do much better.

We are now ready to present the main facts upon which the algorithm rests.

**Lemma 14.** *Let  $x < w \in S_n$  with*

$$x \leftrightarrow (P_x, Q) \quad \text{and} \quad w \leftrightarrow (P_w, Q)$$

*under the Robinson-Schensted correspondence. If the pair  $\{x, w\}$  is a counterexample to the 0-1 Conjecture satisfying the above three minimality properties, then the following eight conditions must hold:*

1.  $D(P_w) \subseteq D(P_x)$ .
2.  $(P_w, Q') > (P_x, Q')$  for all tableaux  $Q'$ .
3. The largest number,  $n - 1$ , sits strictly higher in  $P_w$  than in  $P_x$ .
4. If  $\text{rwd}(P_w)^{-1}(k + 2) < \text{rwd}(P_w)^{-1}(k)$ , then

$$\text{rwd}(P_x)^{-1}(k + 2) < \text{rwd}(P_x)^{-1}(k + 1) < \text{rwd}(P_x)^{-1}(k).$$

5. There do not exist  $L_{i_1}, L_{i_2}, \dots, L_{i_k}$  such that that

$$(14) \quad l(L_{i_k} \cdots L_{i_2} L_{i_1} P_w) - l(L_{i_k} \cdots L_{i_2} L_{i_1} P_x) < l(P_w) - l(P_x).$$

6. There do not exist  $L_{i_1}, L_{i_2}, \dots, L_{i_k}$  satisfying both (14) and

$$(15) \quad l(L_{i_j} L_{i_{j-1}} \cdots L_{i_1} P_w) - l(L_{i_j} L_{i_{j-1}} \cdots L_{i_1} P_x) = l(P_w) - l(P_x), \quad \text{all } j < k.$$

7.  $l(w) - l(x)$  is odd.
8. For no  $i$  are  $0, 1, \dots, i - 1$  in identical positions in  $w$  and  $x$  and are  $\tilde{w}$  and  $\tilde{x}$  tableau words of the same shape.

*Proof.* We give a brief justification for each condition:

1. This follows from Proposition 9.4 and the fact that  $D(P_w) = D_L(\text{rwd}(P_w))$ .
2. Incomparability for some  $Q'$  would imply a contradiction by Theorem 13. So consider the case where  $(P_w, Q') < (P_x, Q')$  for some  $Q'$ . As detailed in [5, Section 5], this implies that there exists some  $Q''$  for which  $w'' \leftrightarrow (P_w, Q'')$  and  $x'' \leftrightarrow (P_x, Q'')$  are related in the *weak* Bruhat-Chevalley order and satisfy  $l(w'') - l(x'') = 1$ . But then, by definition,  $\mu(x'', w'') = 1$ . A contradiction then results by applying Theorem 13.
3. By the previous property, we must have  $\text{cwd}(P_w) \geq \text{cwd}(P_x)$ . This implies that  $n - 1$  must be at least as high in  $P_w$ . If it is the same height, then by Proposition 9.3, you could delete it and get a counterexample in  $S_{n-1}$ .
4. Given minimality property 3, this is equivalent to the first property along with Theorem 13.
5. Knuth transformations preserve left cells; so by Theorem 13, existence would contradict minimality property 2.
6. This is a special case of the previous property.
7. If  $l(w) - l(x)$  is even, then  $\mu(x, w) = 0$ .
8. Otherwise,  $\tilde{x}, \tilde{w}$  lie in the same left cell and afford a smaller counterexample by Proposition 9.3. (Note that  $l(w) - l(x) = l(\tilde{w}) - l(\tilde{x})$ .)

□

We were able to write code to check quickly whether a pair  $(x, w)$  satisfies Properties 1, 3, 4, 7 and 8. It's more time-consuming to check 6. We found it slowest to check Properties 5 and 2.

Since we are working with such large groups, considerable care must be taken to check each of the above eight properties as efficiently as possible. For instance, it is impossibly slow to test property 2 by computing all  $Q'$  tableaux, doing inverse Robinson-Schensted, and checking the Bruhat-Chevalley relations. It is much faster to generate the pairs by doing Knuth transformations and to check for the Bruhat-Chevalley relation by seeing whether the Knuth transformation has destroyed the Bruhat-Chevalley relation which applied before the transformation.

The algorithm used to find a counterexample is as follows:

Step 1. Build up pairs of tableau  $P_x$  and  $P_w$  that satisfy properties 1, 3, 4 and 8 one letter at a time.

Step 2. Successively filter out those pairs not satisfying each of properties 7, 6, 5 and 2.

Step 3. For all remaining pairs  $P_x$  and  $P_w$ , choose  $Q'$  to minimize the length difference between  $x \leftrightarrow (P_x, Q')$  and  $w \leftrightarrow (P_w, Q')$ .

Step 4. Compute  $\mu(x, w)$ . Filter out those pairs for which  $\mu(x, w) \leq 1$ .

No pairs in  $S_{13}$  or below make it through Step 2. In  $S_{14}$  and  $S_{15}$ , none make it through Step 4. But in  $S_{16}$ , the following pair of permutations passes all steps:

$$\begin{aligned} w &= \text{c810d942fa53b6e7}, & l(w) &= 53, \\ x &= \text{54109832dc76baf}, & l(x) &= 32. \end{aligned}$$

The difference in lengths is 21, and the leading coefficient (the coefficient of degree 10) of  $P_{x,w}(q)$  is  $\mu(x, w) = 5$ . The K-L polynomial in its entirety is

$$P_{x,w}(q) = 5q^{10} + 72q^9 + 387q^8 + 1039q^7 + 1610q^6 + 1536q^5 + 931q^4 + 365q^3 + 92q^2 + 14q + 1.$$

This completes the proof of Theorem 1.1.

Exactly one other pair of permutations in  $S_{16}$  passes all the steps of this algorithm, affording a second counterexample to the 0-1 Conjecture:

$$\begin{aligned} w &= \text{ca610fb732d84e95}, & l(w) &= 60, \\ x &= \text{76310cb542a98fed}, & l(x) &= 39, \end{aligned}$$

$$P_{x,w}(q) = 5q^{10} + 56q^9 + 231q^8 + 533q^7 + 776q^6 + 755q^5 + 501q^4 + 226q^3 + 67q^2 + 12q + 1.$$

To give an indication of the computational issues here, in  $S_{16}$  there are  $16! = 20,922,689,888,000$  pairs of tableaux of the same shape, obviously too many to consider. By contrast, Step 1 of our algorithm produces 499,788 pairs. Of these, 246,399 satisfy property 7. Only 672 of these satisfy property 6, and only 288 also satisfy properties 5 and 2. Thus, in searching for a minimal counterexample to the 0-1 Conjecture in  $S_{16}$ , we need to compute  $\mu(x, w)$  at only 288 pairs. The 2 pairs listed above are the only ones having  $\mu(x, w) > 1$ .

Although we have not been able completely to verify the results in this section without the use of computers, we are extremely confident of the truth of Theorem 1.1. The two authors began collaborating after we had independently written programs to compute K-L polynomials, and these programs agree on the values of the polynomials computed above. Only the first author has carried out the process



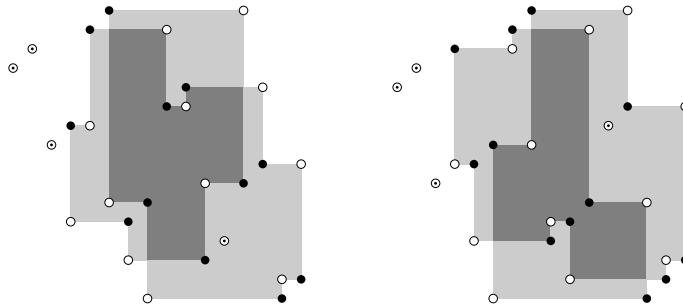


FIGURE 3. The Bruhat pictures for the two minimal counterexamples to the 0-1 Conjecture lying in  $S_{16}$ .

of generating and filtering pairs to produce these counterexamples, but errors in that code would only affect the minimality of our examples. It seems extraordinarily unlikely that our completely independent computations of the K-L polynomials could be incorrect and yet agree. The computer code used in the proof of Theorem 1.1 (along with Java code for computing K-L polynomials) can be found at [11].

#### 4. COMBINATORIAL PROOF OF THEOREM 1.2

The previous section describes a counterexample showing that for large enough  $n$ , labeled graphs are required for Kazhdan and Lusztig's description of the irreducible representations of  $S_n$ . In this section, we remove the condition that the two permutations lie in the same left cell. Such a counterexample is less interesting representation-theoretically, but it can be carried out entirely by hand, and it should lend insight into how and when  $\mu(x, w)$  can be greater than 1. The counterexample we present was arrived at by close examination (using the "Bruhat pictures" of [2]) of the counterexample presented in the previous section.

We begin our proof of Theorem 1.2 by first calculating several intermediate KL polynomials. The main tools in the proof are the defining recurrence relation (5) and parts 1 and 3 of Proposition 9. For each application of (5), there are usually several choices of the generator  $s$ . While our choices for  $s$  may seem *ad hoc*, they are actually carefully made both to maximize the number of applications of Proposition 9.3 we can make and to simplify the calculation of the resulting  $\Theta$ 's.

**Lemma 15.** *The following equalities hold:*

1.  $P_{1032,3120} = 1 + q$ .
2.  $P_{0213,2301} = 1 + q$ .
3.  $P_{315042,534120} = 1 + 3q + q^2$ .
4.  $P_{3106542,6345120} = 1 + 4q + 4q^2 + q^3$ .

In the rest of this paper, for layout reasons, we sometimes write  $P_w^x$  for  $P_{x,w}$ .

*Proof.* The first two equalities can be shown immediately using (5) or they can be found in [2]. For the third equality, we begin by expanding using (5) with  $s = s_4$ :

$$(16) \quad P_{534120}^{315042} = qP_{534102}^{315042} + P_{534102}^{315024} - \Theta_{(\cdot, s_4)}[315042, 534102].$$

Consider  $\Theta_{(\cdot, s_4)}[315042, 534102]$ . It is clear that there are no  $s_4$ -coatomic elements. By Lemma 5, for any  $z \in [315042, 534102]$ , we need  $z(5) = 2$ . And for  $z$  to be  $s_4$ -flush, we need  $D_{\mathbb{R}}(z) \supseteq \{s_0, s_2, s_3, s_4\}$ . But these conditions cannot simultaneously be satisfied along with  $z(0) \geq 3$  (necessary for  $z \geq 315042$ ). So  $\theta_{(\cdot, s_4)}[315024, 534102] = \emptyset$  and  $\Theta_{(\cdot, s_4)}[315024, 534102] = 0$ . Therefore,

$$\begin{aligned} P_{534120}^{315042} &= qP_{42310}^{21403} + P_{534102}^{315204} && \text{(Proposition 9, parts 1 \& 3)} \\ &= qP_{42310}^{21430} + P_{42301}^{20413} && \text{(Proposition 9, parts 1 \& 3)} \\ &= qP_{3120}^{1032} + \left( qP_{24301}^{20413} + P_{24301}^{02413} - \Theta_{(\cdot, s_0)}[20413, 24301] \right). \end{aligned}$$

(The last equality follows from Proposition 9.3 and (5) with  $s = s_0$ .)

It is clear that there are no  $s_0$ -coatomic elements. If  $z \in [20413, 24301]$ , then it follows from Lemma 5 that  $z(0) = 2$ . But for  $z$  to be  $s_0$ -flush we need  $D_{\mathbb{R}}(z) \supseteq \{s_0, s_1, s_2\}$ . These two conditions cannot simultaneously be satisfied. Therefore,  $\theta_{(\cdot, s_0)}[20413, 24301] = \emptyset$  and  $\Theta_{(\cdot, s_0)}[20413, 24301] = 0$ . Therefore,

$$\begin{aligned} P_{534120}^{315042} &= q(1+q) + (qP_{3201}^{0312} + P_{24301}^{04213}) && \text{(Lemma 15.1; Proposition 9, parts 1 \& 3)} \\ &= q(1+q) + (q \cdot 1 + P_{2301}^{0213}) && \text{(Theorem 6.1; Proposition 9, parts 1 \& 3)} \\ &= q(1+q) + (q \cdot 1 + (1+q)) && \text{(Lemma 15.2)} \\ &= 1 + 3q + q^2. \end{aligned}$$

For the fourth equality, we set  $s = s_3$  in (5), and utilize Proposition 9.1:

$$P_{6345120}^{3106542} = (1+q)P_{6341520}^{3106542} - \Theta_{(\cdot, s_3)}[3106542, 6341520].$$

There are no  $s_3$ -coatomic elements in  $[3106542, 6341520]$ . If  $z$  is in this interval, then by Lemma 5,  $z(4) = 5$ . For  $z$  to be  $s_3$ -flush, it must satisfy  $D_{\mathbb{R}}(z) \supseteq \{s_0, s_2, s_3, s_4, s_5\}$ . As these conditions cannot be simultaneously satisfied, we conclude that  $\theta_{(\cdot, s_3)}[3106542, 6341520] = \emptyset$ ; hence  $\Theta_{(\cdot, s_3)}[3106542, 6341520] = 0$ . Therefore,

$$\begin{aligned} P_{6345120}^{3106542} &= (1+q)P_{6341520}^{3106542} \\ &= (1+q)P_{534120}^{315042} && \text{(Proposition 9, parts 1 \& 3)} \\ &= (1+q)(1+3q+q^2) && \text{(Lemma 15.3)} \\ &= 1 + 4q + 4q^2 + q^3. \end{aligned}$$

□

In addition to the above KL polynomials, we also need to compute several  $\Theta$ 's.

**Lemma 16.** *The following equalities hold:*

1.  $\Theta_{(s_2 \cdot)}[32170654, 72561340] = q^4$ .
2.  $\Theta_{(\cdot, s_4)}[321087654, 835617240] = 0$ .
3.  $\Theta_{(\cdot, s_3)}[4321098765, 9461782350] = q^4(1+q)$ .

	$z$	$\tilde{z}$	$\tilde{v}$
$z_1$	32170654	2160543	6451230
$z_2$	32175640	10423	42301
$z_3$	32751640	0312	3120

TABLE 1. Cases for Lemma 16.1.

*Proof.* It is easily checked that there are no coatomic elements for any of the three above cases. Hence, in the following, we will assume that  $z$  is  $s$ -flush.

1. Let  $x = 32170654$  and  $v = s_2w = 72561340$ . To find the elements of  $\omega_{(s_2)}[x, v]$  is straightforward but tedious. If  $z \in \omega_{(s_2)}[x, v]$ , then helpful facts about  $z$  include
  - (a)  $\{s_0, s_1, s_2, s_4, s_6\} \subseteq D_L(z)$ .
  - (b)  $\{s_0, s_3, s_6\} \subseteq D_R(z)$ .
  - (c)  $z(1) = 2$ , which combined with 1a implies that  $z(0) = 3$ .
  - (d)  $z(2) \in \{1, 5, 7\}$ .
  - (e) To have  $z \in [x, v]$  requires that  $z^{-1}(1) \in \{2, 3, 4\}$ , that  $z^{-1}(7) \in \{2, 3\}$ , and that  $z^{-1}(6) \in \{3, 4, 5\}$ .

Armed with these facts, it is not hard to find the nine elements of  $\omega_{(s_2)}[x, v]$ . Of these nine, only three have an odd length difference with respect to  $v$  (an even length difference with respect to  $w$ ); only these three, which are shown in Table 1, can contribute to  $\Theta_{(s_2)}[32170654, 72561340]$ .

We know from Lemma 15.4 and Proposition 9, parts 2 and 3 that  $P_{z_1, v} = 1 + 4q + 4q^2 + q^3$ . As  $l(v) - l(z_1) = 7$ ,  $\mu(z_1, v) = 1$ . Finally, since  $P_{z_1, z_1} = 1$ , the only non-zero term of  $\Theta_{(s_2)}[x, v]$  is  $1 \cdot q^4 \cdot 1 = q^4$ .

2. If  $z \in [321087654, 835617240]$  then by Lemma 5,  $z(5) = 7$ . And if  $z \in \omega_{(s_4)}[321087654, 835617240]$ , then  $D_R(z) \supseteq \{s_0, s_3, s_4, s_5, s_7\}$ . These two conditions cannot be satisfied simultaneously.
3. Let  $x = 4321098765$  and  $v = ws_3 = 9461782350$ . To find the elements of  $\omega_{(s_3)}[x, v]$  is again straightforward but still more tedious. If  $z \in \omega_{(s_3)}[x, v]$ , then
  - (a)  $D_R(z) \supseteq \{s_0, s_2, s_3, s_5, s_8\}$ .
  - (b)  $D_L(z) \supseteq \{s_0, s_3, s_5, s_8\}$ .
  - (c)  $z(3) = 1$ ;  $z(4) = 0$ .
  - (d)  $z^{-1}(8) \leq 6$ ;  $z^{-1}(7) \leq 7$ .
  - (e)  $z^{-1}(9) \in \{0, 2, 5\}$ .
  - (f)  $z^{-1}(8) \in \{5, 6\}$ .
  - (g)  $z(9) \in \{2, 3, 5\}$ .

These observations let us generate the 34 elements of  $\omega_{(s_3)}[x, v]$ . Since  $l(v) = 30$ , we can only have  $\mu(z, v) \neq 0$  if  $l(z)$  is odd. In Table 3, we list the seventeen of these  $z$  with an odd length difference with respect to  $v$  along with the corresponding  $\tilde{z}$  and  $\tilde{v}$ .

By Proposition 9.4, we ascertain that the only  $z$  in the above table for which we might have  $\mu(z, v) \neq 0$  is  $z_5 = 6421098753$ . By Proposition 9.3 and Lemma 15.4,  $P_{z_5, v} = 1 + 4q + 4q^2 + q^3$ . As  $l(v) - l(z_5) = 7$ ,  $\mu(z_5, v) = 1$ . Proposition 9.3, along with Lemma 15.1, shows that  $P_{x, z_5} = 1 + q$ . The only non-zero contribution to the sum in (5) is therefore  $1 \cdot q^4 \cdot (1 + q) = q^4(1 + q)$ , as desired.

	$z$	$\tilde{z}$	$\tilde{v}$
$z_1$	4371098265	326087154	8356712410
$z_2$	4371098652	32507641	73456120
$z_3$	4391087265	32706154	73561240
$z_4$	4391087652	3260541	6345120
$z_5$	6421098753	3106542	6345120
$z_6$	6471098352	230541	523410
$z_7$	6491082753	350142	534120
$z_8$	6491087352	24031	42310
$z_9$	7431098265	52076143	74561230
$z_{10}$	7431098652	4206531	6345120
$z_{11}$	7461098253	305412	534120
$z_{12}$	9421083765	102543	451230
$z_{13}$	9421086753	10342	34120
$z_{14}$	9431087265	205143	451230
$z_{15}$	9431087652	20431	34120
$z_{16}$	9461083752	0231	3120
$z_{17}$	9461087253	0312	3120

TABLE 2. Cases for Lemma 16.3.

□

*Proof of Theorem 1.2.* By (5) with  $s = s_3$  and Proposition 9.1,

$$(17) \quad P_{\frac{4321098765}{9467182350}} = (1+q)P_{\frac{4321098765}{9461782350}} - \Theta_{(\cdot, s_3)}[4321098765, 9461782350].$$

By Lemma 16.3,  $\Theta_{(\cdot, s_3)}[4321098765, 9461782350] = q^4(1+q)$ . Using Proposition 9.3, we can therefore rewrite (17) as:

$$(18) \quad P_{\frac{4321098765}{9467182350}} = (1+q)P_{\frac{321087654}{835671240}} - q^4(1+q).$$

Expanding using (5) with  $s = s_4$  and applying Proposition 9.1 and Lemma 16.2, we get

$$(19) \quad P_{\frac{4321098765}{9467182350}} = (1+q) \left( (1+q)P_{\frac{321087654}{835617240}} - 0 \right) - q^4(1+q).$$

By Proposition 9, parts 1 and 3, this can be rewritten:

$$(20) \quad P_{\frac{4321098765}{9467182350}} = (1+q)(1+q)P_{\frac{32170654}{73561240}} - q^4(1+q)$$

$$(21) \quad = (1+q)^2 \left( (1+q)P_{\frac{32170654}{72561340}} - q^4 \right) - q^4(1+q).$$

The second follows from the first by the left-hand version of (5) with  $s = s_2$  and Lemma 16.1. Simplifying according to Proposition 9.3, we get

$$(22) \quad P_{\frac{4321098765}{9467182350}} = (1+q)^2 \left( (1+q)P_{\frac{2160543}{6451230}} - q^4 \right) - q^4(1+q)$$

$$(23) \quad = (1+q)^2 \left( (1+q)(1+4q+4q^2+q^3) - q^4 \right) - q^4(1+q)$$

$$(24) \quad = 1 + 7q + 19q^2 + 26q^3 + 17q^4 + 4q^5.$$

Going from (22) to (23) uses Lemma 15.4 and Proposition 9.2. As  $l(9467182350) - l(4321098765) = 11$ , this completes the proof of the theorem. □

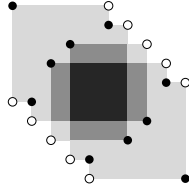


FIGURE 4. The Bruhat picture corresponding to the counterexample to the 0-1 Conjecture lying in  $S_{10}$ .

**Remark 17.**  $\mu(x, w)$  can, in fact, be 2 or 3, though we have not yet found any examples in groups smaller than  $S_{14}$ .

$$w = 789ab0cd123456, \quad l(w) = 47,$$

$$x = 0759321cba486d, \quad l(x) = 32,$$

$$P_{x,w}(q) = 2q^7 + 111q^6 + 693q^5 + 1292q^4 + 908q^3 + 257q^2 + 29q + 1.$$

$$w = 789ab0cd123456, \quad l(w) = 47,$$

$$x = 0784321cba956d, \quad l(x) = 32,$$

$$P_{x,w}(q) = 3q^7 + 124q^6 + 716q^5 + 1346q^4 + 960q^3 + 263q^2 + 29q + 1.$$

### 5. COMPUTATIONS ON REMAINING CONJECTURES

In Section 1, we mention the conjectures that the L-S graph obtained by taking the trivial edges in a left cell and adding all the edges obtained from these by Knuth relations might be the same as the K-L graph, or at least that this graph might give rise to a representation of  $S_n$ . That the first of these conjectures is false follows at once from the counterexample in Theorem 1.1, since every edge of the L-S graph has weight 1. Thus, this conjecture must be false starting in  $S_{16}$ . In fact, a computer search for counterexamples to this conjecture, carried out in a manner analogous to that in Section 3, shows that the first counterexamples to this conjecture appear in  $S_{14}$ . We obtained these counterexamples before the counterexamples to the 0-1 Conjecture, and their existence inspired us to continue searching for edges of weight greater than 1.

A typical edge in  $S_{14}$  that cannot be obtained from any trivial edge by a sequence of Knuth relations is that joining  $w = db630c7418295a$  and  $x = 76530db4192c8a$ .

Not only is the L-S graph not the same as the K-L graph, but the L-S graph fails to give rise to a representation of  $S_n$ , again starting at  $n = 14$ . This has been shown by the first author via computer calculations. In particular, take the tableaux whose column words are the permutations in the counterexample above,

$$P_w = \begin{array}{|c|c|c|c|} \hline d & & & \\ \hline b & c & & \\ \hline 6 & 7 & & \\ \hline 3 & 4 & 8 & 9 \\ \hline 0 & 1 & 2 & 5 & a \\ \hline \end{array} \quad \text{and} \quad P_x = \begin{array}{|c|c|c|c|c|} \hline 7 & & & & \\ \hline 6 & d & & & \\ \hline 5 & b & & & \\ \hline 3 & 4 & 9 & c & \\ \hline 0 & 1 & 2 & 8 & a \\ \hline \end{array} .$$

Choose any tableau  $Q$  of the same shape as  $P_x$  and  $P_w$  and let

$$x \leftrightarrow (P_x, Q) \text{ and } w \leftrightarrow (P_w, Q)$$

under the Robinson-Schensted correspondence. For an adjacent transposition  $s$ , let  $A(s)$  be the matrix obtained by following the recipe of Kazhdan and Lusztig starting with the L-S graph, and let  $B(s)$  be the matrix obtained by following the recipe of Kazhdan and Lusztig starting with the K-L graph. Since the  $B(s)$  generate a representation, we know that for every commuting  $s$  and  $t$  we have the equality of the matrix entries

$$(25) \quad (B(s)B(t))_{w,tx} = (B(t)B(s))_{w,tx}.$$

If the  $A(s)$  also generate a representation, then we should also have

$$(26) \quad (A(s)A(t))_{w,tx} = (A(t)A(s))_{w,tx}$$

for all choices of commuting  $s$  and  $t$ .

If  $s = s_4$  and  $t = s_b$ , then the edge between  $x$  and  $w$ , present in the K-L graph and absent in the L-S graph, makes a contribution to the left hand side of (25) via the term

$$B(s)_{w,x}B(t)_{x,tx} = 1 \cdot 1 = 1.$$

The corresponding term on the left hand side of (26) is

$$A(s)_{w,x}A(t)_{x,tx} = 0 \cdot 1 = 0.$$

It is not obvious that this edge forces different contributions to the right hand sides of (25) and (26). This suggests that for these values of  $x$  and  $w$  and  $s$  and  $t$ , (26) might well be false.

The representation here has dimension 68,640, and finding each matrix entry involves computing a K-L polynomial; but the matrices are sufficiently sparse that nearly every term in the matrix products in (26) is obviously 0. It is therefore not difficult to show by computer that for this  $x, w, s, t$  we have

$$(A(s)A(t))_{w,tx} = 0 \neq 1 = (A(t)A(s))_{w,tx}.$$

Thus, this conjecture, like all those mentioned here, is also false.

## 6. ACKNOWLEDGEMENTS

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